PRACTICE PAPERS

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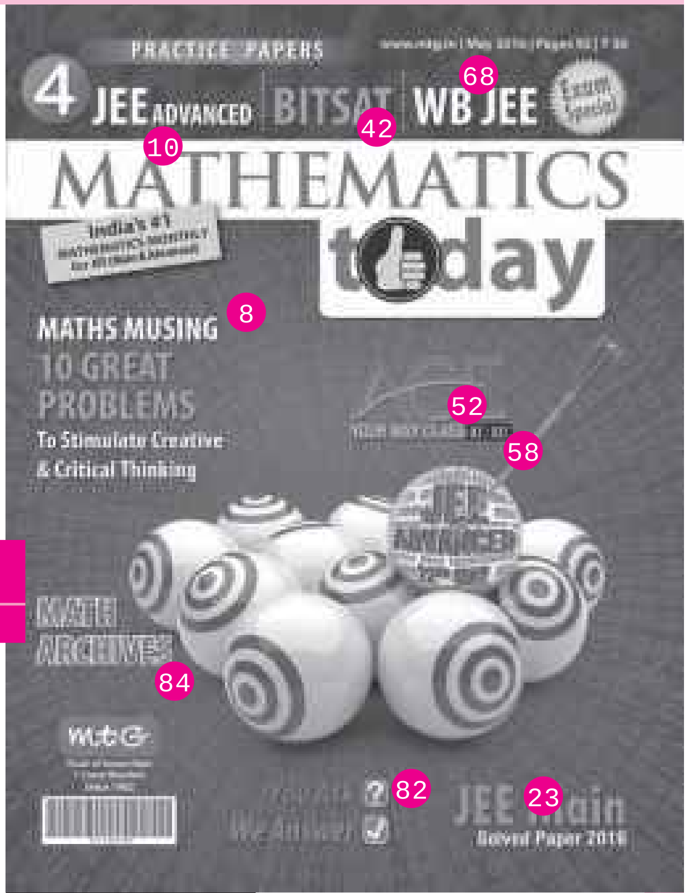

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Editor : Anil Ahlawat

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 161

JEE MAIN

1. $\int_1^{16} \tan^{-1}\sqrt{\sqrt{x}-1}\,dx =$

(a) $\frac{8\pi}{3}+\sqrt{3}$ (b) $\frac{8\pi}{3}-\sqrt{3}$

(c) $\frac{16\pi}{3}-2\sqrt{3}$ (d) $\frac{16\pi}{3}+2\sqrt{3}$

2. If $x = \sin\alpha$, $y = \sin\beta$, $z = \sin(\alpha+\beta)$, then $\cos(\alpha+\beta) =$

(a) $\frac{x^2+y^2+z^2}{2xy}$ (b) $\frac{x^2+y^2-z^2}{xy}$

(c) $\frac{z^2-x^2-y^2}{2xy}$ (d) $\frac{z^2-x^2-y^2}{xy}$

3. The vertices of a triangle are $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$. If a, b, -111 are the direction ratios of the line joining the orthocentre and circumcentre, then $a + b =$

(a) 5 (b) 10 (c) 15 (d) 25

4. Let N be the number of triples (a, b, c) of positive integers such that $a > b > c$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$. Then N is divisible by

(a) 2 (b) 9 (c) 5 (d) 7

5. If $(1+px+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then $\sum_{r=0}^{2n}(2r+1)a_r =$

(a) $(p+2)^n$ (b) $(2p+1)(p+2)^n$

(c) $(2n+1)(p+2)^n$ (d) $(p+2)^{n+1}$

JEE ADVANCED

6. Let $f(x)$ be the fourth degree polynomial such that $f'(0) = -6, f'(2) = 6$ and $\lim_{x\to 1}\frac{f(x)}{(x-1)^2}=1$

Then, the length of sub-tangent of the curve $y = f(x)$, where it cuts the y-axis is

(a) 1 (b) 1/2 (c) 1/3 (d) 1/4

COMPREHENSION

In a triangle ABC, let $A : B : C = 1 : 2 : 4$

7. $\tan A \tan B + \tan B \tan C + \tan C \tan A =$

(a) -3 (b) -5

(c) -7 (d) -9

8. $\sec^2 A + \sec^2 B + \sec^2 C =$

(a) 5 (b) 21

(c) 24 (d) 27

INTEGER MATCH

9. Let $f(x)$ be the ratio of two quadratic polynomials. If $f(0) = 6$ and $f(x)$ assumes turning values 3 and 4 at $x = 2$ and $x = -2$ respectively, then $f(1) = \frac{m}{n}$, where m and n are co-prime positive integers and $m - n$ is

MATRIX MATCH

10.

(a)	If 4 dice are rolled, the probability of getting the sum 10, is	(p)	$\frac{7}{15}$
		(q)	$\frac{2}{7}$
(b)	If 10 men are sitting in a row, the probability of choosing 3 of them so that no two are from adjacent seats is	(r)	$\frac{8}{15}$
(c)	Triangles are formed with vertices of a regular octagon. If a triangle is chosen at random, the probability that it does not have any side common with the octagon, is	(s)	$\frac{5}{27}$
		(t)	$\frac{5}{81}$

See Solution set of Maths Musing 160 on page no. 67

JEE Advanced

PRACTICE PAPER 2016

SECTION-1

Only One Option Correct Type

This section contains 12 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. If a_1, a_2, a_3, ..., a_n are 'n' positive numbers, then minimum value of $\frac{a_1}{a_2}+2\frac{a_2}{a_3}+3\frac{a_3}{a_4}+4\frac{a_4}{a_5}+.....+n\frac{a_n}{a_1}$ is always greater than

(a) $n^2(n!)^{1/n}$ (b) $n!$
(c) $n(n!)^{1/n}$ (d) $n(n!)$

2. If $f(x)=\frac{1}{x^2-4x+3}$, $g(x)=\frac{1}{x-2}$, then number of points where $f(g(x))$ is discontinuous is

(a) 3 (b) 2 (c) 1 (d) 0

3. The interval at x for which $\frac{\sin^3 x}{1+\cos x}+\frac{\cos^3 x}{1-\sin x}\geq 0$ is

(a) $n\pi < x < (n+1)\pi$, $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

(b) $2n\pi+\frac{\pi}{4}\leq x\leq 2n\pi+\frac{5\pi}{4}$,

$x\neq(4n+1)\frac{\pi}{2}$, $(2n+1)\pi$, $n\in Z$

(c) $2n\pi < x < (2n+1)\pi$, $n \in Z$
(d) none of these

4. A plane passes through $(1,-2,1)$ and is perpendicular to two planes $2x-2y+z=0$ and $x-y+2z=4$. The distance of plane from the point $(0, 2, 2)$ is

(a) $\frac{3}{\sqrt{2}}$ units (b) $4\sqrt{2}$ units
(c) $3\sqrt{2}$ units (d) $2\sqrt{2}$ units

5. $\int_0^{\frac{11}{3}\pi}\sqrt{1+\cos 2x}\,dx$ is equal to

(a) $\frac{11\sqrt{3}}{2}$ (b) $8\sqrt{2}-\sqrt{\frac{3}{2}}$

(c) $8\sqrt{2}-\frac{\sqrt{3}}{2}$ (d) $\frac{22}{3}$

6. Consider the vectors $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+\hat{k}$. A parallelogram is constructed such that its diagonals are along $2\vec{a}-\vec{b}$ and $\vec{a}-2\vec{b}$. The area of the parallelogram is

(a) $9\sqrt{5}$ sq. units (b) $\frac{9\sqrt{10}}{2}$ sq. units

(c) $9\sqrt{10}$ sq. units (d) $18\sqrt{5}$ sq. units

7. Let $\frac{\tan\left(\frac{\pi}{4}+\alpha\right)}{5}=\frac{\tan\left(\frac{\pi}{4}+\beta\right)}{3}=\frac{\tan\left(\frac{\pi}{4}+\gamma\right)}{2}$. Then $12\sin^2(\alpha-\beta)+15\sin^2(\beta-\gamma)-7\sin^2(\gamma-\alpha)$ is equal to

(a) $-1/2$ (b) $1/2$ (c) 1 (d) 0

8. The angle of intersection of the curves $y=(1+x)^{\cos x}+\sin x$ and $y=\frac{1}{2}(x^2+x+2)$ at (h, k) where h and k are integers, is θ, then $\tan\theta$ equals

(a) 3/4 (b) 4/3 (c) 1 (d) 0

9. The area of the triangle formed by the line $x+y=2$ and angle bisectors of the pair of straight

lines $x^2 - y^2 + 2y = 1$ is a and coordinates of orthocentre of triangle are (b, c), then $4a + b + c$ is equal to

(a) 0 (b) 4 (c) 2 (d) 3

10. If z is the complex 5^{th} root of unity, then

$$\frac{z^4}{z^4+1}+\frac{z}{1+z}+\frac{z^2}{z^2+1}+\frac{z^4}{z+z^4} \text{ is}$$

(a) 2 (b) 0 (c) 4 (d) 1

11. If a tangent of slope 3 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

(a) 2 (b) 3 (c) 4 (d) 6

12. If $F(x)=\int_0^x f(t)\cdot dt$ and $F(x^2) = x^2(1 + x)$, then the value of $f(4) = n$. Then unit place of $(985697)^{n^2}$ is

(a) 3 (b) 7 (c) 1 (d) 9

SECTION-2

One or More Option Correct Type

This section contains 6 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE are correct.

13. A curve passes through (5, 4) and the slope of tangent at (x, y) is $\frac{1-x}{y-1}$, then

(a) The area of triangle formed by tangent and normal at (5, 4) and the x-axis is 50/3 sq. units

(b) the area of triangle formed by tangent and normal at (5, 4) and the x-axis is 86/3 sq. units

(c) The length of perpendicular from (0, 0) to tangent at (5, 4) is 32/5 units

(d) The length of perpendicular from (0, 0) to tangent at (5, 4) is 16/5 units

14. Let $f(x)$ be an integrable function in $0 \le x \le 1$ and suppose

$$\int_0^1 f(x)\,dx=0,\ \int_1^0 xf(x)dx=0, \ldots\ldots, \int_0^1 x^{n-1}f(x)dx=0 \text{ and}$$

$$\int_0^1 x^n f(x)dx=1, \text{ then}$$

(a) $\int_0^1 (2x-1)^n f(x)\,dx=0$

(b) $\int_0^1 (2x-1)f(x)\,dx=2^n$

(c) Value of $\int_0^1 \left(x-\frac{1}{2}\right)^n f(x)\,dx=1$

(d) Value of $\int_0^1 \left(x-\frac{1}{2}\right)^n f(x)\,dx=0$

15. Let $f(x)=\frac{(x-5)^p}{\ln(\cos(x-5))^q}$, $4<x<6$; p, q are integers such that $p \ne 0$, $q > 0$. If $\lim_{x\to 5^+} f(x)=-1$, then

(a) $p = 2$ (b) $p = -1$

(c) $q = 1$ (d) $q = 2$

16. If the sides a, b, c of a triangle ABC satisfy the equation $313a^2 + 25b^2 + 25c^2 = 10a(12b + 13c)$, then

(a) ABC is right angled triangle

(b) $\angle A=\cos^{-1}\left(\frac{12}{13}\right)$

(c) $a : b = 5 : 12$

(d) $\angle C=\cos^{-1}\left(\frac{5}{13}\right)$

17. If the straight line $3x + 4y = 24$ intersects the axes at P and Q and straight line $4x + 3y = 24$ at R and S, then point $PQRS$ lies on

(a) circle (b) parabola

(c) ellipse (d) hyperbola

18. The function $f(x)$ satisfying $(f(x))^2 - 4f(x)f'(x) + (f'(x))^2 = 0$ is given by

(a) $ke^{(2+\sqrt{3})x}$ (b) $ke^{(2-\sqrt{3})x}$

(c) $ke^{(2+\sqrt{6})x}$ (d) $ke^{(2-\sqrt{6})x}$

SECTION-3

Linked Comprehension Type

This section contains one paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices, (a), (b), (c) and (d), out of which ONLY ONE is correct.

In a problem of differentiation of $\frac{f(x)}{g(x)}$, one student writes the derivative of $\frac{f(x)}{g(x)}$ as $\frac{f'(x)}{g'(x)}$ and he finds the correct result if $g(x) = x^2$ and $\lim_{x\to\infty} f(x)=4$. A circle 'C'

of minimum radius is drawn which intersects both the curves $y = f(x)$ and $y = g(x)$ at two points at which they intersect. Let 'P' be a point on $y = g(x)$.

19. $\lim\limits_{x\to\infty}\left(\dfrac{f(x)}{4}\right)^{\frac{x^2}{2x+1}}$

(a) e (b) 1 (c) e^{-2} (d) e^2

20. $\sum\limits_{r=1}^{n+2}\dfrac{r^2}{f(r)}$ is

(a) $\dfrac{n(n+1)(2n+1)}{24}$ (b) $\dfrac{n(n+1)(2n+1)}{6}$

(c) $\dfrac{1}{4}+\dfrac{n(n+1)(2n+1)}{24}$ (d) $1+\dfrac{n(n+1)(2n+1)}{24}$

21. Coordinates of 'P' at which tangent to $y = g(x)$ is parallel to common chord of $y = f(x)$ and $y = g(x)$ are

(a) (1/2, 1/4) (b) (2, 4)

(c) (4, 16) (d) (0,0)

SECTION-4

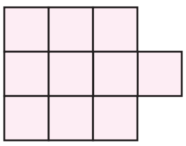
Integer Value Correct Type

This section contains 4 questions. The answer to each of the question is a single digit integer, ranging from 0 to 9.

22. The minimum distance between the curves $y=\sqrt{2-x^2},\ \ y=\dfrac{9}{x};\ x,\ y>0$ is k, then k^2 equals ________.

23. If $f: [-1, 1] \to R$ be a continuous function satisfying $f(2x^2 - 1) = (x^3 + x) f(x)$, then find $\lim\limits_{x\to 0}\dfrac{f(\cos x)}{\sin x}$.

24. The number of ways in which 7 different letters can be arranged in the figure given below such that no row is empty is $59k \cdot 7!$



Then k is ________.

25. A nine digit number is formed from 1, 2, 3, 4, 5 such that the product of all digits is always 1920. The total number of ways is $393({}^9P_r)$, where the value of r is ________.

SOLUTIONS

1. (c) : $\dfrac{\dfrac{a_1}{a_2}+2\dfrac{a_2}{a_3}+3\dfrac{a_3}{a_4}+\ldots+n\dfrac{a_n}{a_1}}{n}\geq(1\cdot2\cdot3\ldots..n)^{1/n}$

$$\Rightarrow \frac{a_1}{a_2}+2\frac{a_2}{a_3}+\ldots+n\frac{a_n}{a_1}\geq n(n!)^{1/n}$$

2. (a) : $f(g(x)) = f\left(\dfrac{1}{x-2}\right),\ x\neq 2$

$$=\frac{1}{\dfrac{1}{(x-2)^2}-\dfrac{4}{(x-2)}+3}=\frac{(x-2)^2}{1-4(x-2)+3(x-2)^2}$$

$$x\neq 3,\ \frac{7}{3}$$

$\therefore$ Number of points of discontinuity = 3

3. (d): $\sin x(1 - \cos x) + \cos x(1 + \sin x) \geq 0$, $\cos x \neq -1$, $\sin x \neq 1$ $\Rightarrow$ $\sin x + \cos x \geq 0$

$$\Rightarrow \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\geq 0$$

$$0\leq x+\frac{\pi}{4}<\pi \Rightarrow -\frac{\pi}{4}\leq x<\frac{3\pi}{4},\ x\neq\frac{\pi}{2}$$

$$2n\pi-\frac{\pi}{4}\leq x\leq 2n\pi+\frac{3\pi}{4},\ x\neq\frac{\pi}{2}$$

4. (a) : Equation of plane passing through (1, –2, 1) is $a(x - 1) + b(y + 2) + c(z - 1) = 0$

Also, $2a - 2b + c = 0$

$a - b + 2c = 0$

$$\therefore \frac{a}{-3}=\frac{b}{-3}=\frac{c}{0}$$

$\therefore$ Plane is $(x - 1) + (y + 2) = 0$

$\Rightarrow$ $x + y + 1 = 0$

Length of perpendicular from (0, 2, 2) to the plane

$$=\left|\frac{0+2+1}{\sqrt{2}}\right|=\frac{3}{\sqrt{2}}\text{ units}$$

5. (b): $I=\displaystyle\int_0^{\frac{11\pi}{3}}\sqrt{1+\cos 2x}\,dx$

$$=\sqrt{2}\left\{\int_0^{3\pi}|\cos x|\,dx+\int_{3\pi}^{3\pi+\frac{2\pi}{3}}|\cos x|\,dx\right\}$$

$$=\sqrt{2}\left\{3\int_0^{\pi}|\cos x|\,dx+\int_0^{\frac{2\pi}{3}}|\cos x|\,dx\right\}$$

$$=\sqrt{2}\left\{\int_0^{\pi/2}6\cos x\,dx+\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{2\pi/3}-\cos x\,dx\right\}$$

$$= \sqrt{2}\left\{7 - \left(\frac{\sqrt{3}}{2} - 1\right)\right\} = 8\sqrt{2} - \sqrt{\frac{3}{2}}$$

6. (b): $2\vec{a} - \vec{b} = 2(2\hat{i} + \hat{j} - 2\hat{k}) - (3\hat{i} - \hat{j} + \hat{k})$

$$= \hat{i} + 3\hat{j} - 5\hat{k}$$

$$\vec{a} - 2\vec{b} = (2\hat{i} + \hat{j} - 2\hat{k}) - 2(3\hat{i} - \hat{j} + \hat{k})$$

$$= -4\hat{i} + 3\hat{j} - 4\hat{k}$$

$$(2\vec{a} - \vec{b}) \times (\vec{a} - 2\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -5 \\ -4 & 3 & -4 \end{vmatrix} = 3\hat{i} + 24\hat{j} + 15\hat{k}$$

The area of the parallelogram

$$= \frac{1}{2}|(2\vec{a} - \vec{b}) \times (\vec{a} - 2\vec{b})| = \frac{1}{2}\sqrt{9 + 576 + 225}$$

$$= \frac{1}{2}\sqrt{810} = \frac{9\sqrt{10}}{2} \text{ sq. units}$$

7. (d): $\dfrac{\tan\left(\frac{\pi}{4} + \alpha\right)}{\tan\left(\frac{\pi}{4} + \beta\right)} = \dfrac{5}{3}$

Applying componendo-dividendo rule, we get,

$$\frac{\sin\left(\frac{\pi}{4} + \alpha + \frac{\pi}{4} + \beta\right)}{\sin\left(\frac{\pi}{4} + \alpha - \frac{\pi}{4} - \beta\right)} = \frac{5+3}{5-3} \Rightarrow \frac{\cos(\alpha + \beta)}{\sin(\alpha - \beta)} = 4$$

$\Rightarrow 4\sin(\alpha - \beta) = \cos(\alpha + \beta)$ ... (i)

Now, $\dfrac{\tan\left(\frac{\pi}{4} + \beta\right)}{\tan\left(\frac{\pi}{4} + \gamma\right)} = \dfrac{3}{2}$

$$\Rightarrow \frac{\sin\left(\frac{\pi}{4} + \beta + \frac{\pi}{4} + \gamma\right)}{\sin\left(\frac{\pi}{4} + \beta - \frac{\pi}{4} - \gamma\right)} = \frac{3+2}{3-2} \Rightarrow \frac{\cos(\beta + \gamma)}{\sin(\beta - \gamma)} = 5$$

$\Rightarrow 5\sin(\beta - \gamma) = \cos(\beta + \gamma)$ ... (ii)

and $\dfrac{\tan\left(\frac{\pi}{4} + \gamma\right)}{\tan\left(\frac{\pi}{4} + \alpha\right)} = \dfrac{2}{5} \Rightarrow \dfrac{\sin\left(\frac{\pi}{4} + \gamma + \frac{\pi}{4} + \alpha\right)}{\sin\left(\frac{\pi}{4} + \gamma - \frac{\pi}{4} - \alpha\right)} = \dfrac{2+5}{2-5}$

$$\Rightarrow \frac{\cos(\gamma + \alpha)}{\sin(\gamma - \alpha)} = \frac{7}{-3}$$

$\Rightarrow -\dfrac{7}{3}\sin(\gamma - \alpha) = \cos(\gamma + \alpha)$ ... (iii)

Now multiplying (i) by 3 sin $(\alpha - \beta)$, (ii) by 3 sin$(\beta - \gamma)$, (iii) by 3 sin $(\gamma - \alpha)$ and adding, we get 12 $\sin^2(\alpha - \beta)$ + 15 $\sin^2(\beta - \gamma)$ – 7$\sin^2(\gamma - \alpha)$

$$= \frac{3}{2}\{(\sin 2\alpha - \sin 2\beta) + (\sin 2\beta - \sin 2\gamma) + (\sin 2\gamma - \sin 2\alpha)\} = 0$$

8. (a): $y = (1+x)^{\cos x} + \sin x,\ y = \dfrac{1}{2}(x^2 + x + 2)$

Clearly at $x = 0$ both curves intersect and the point of intersection is (0, 1)

$y = (1 + x)^{\cos x} + \sin x$

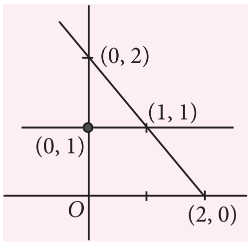
$$\Rightarrow y' = (1+x)^{\cos x}\left\{\frac{\cos x}{1+x} - \sin x \cdot \ln(1+x)\right\} + \cos x$$

$\Rightarrow m_1 = 2$

and for $y = \dfrac{1}{2}(x^2 + x + 2) \Rightarrow y' = \dfrac{1}{2}(2x+1)$

$$\Rightarrow m_2 = \frac{1}{2} \quad \therefore \tan\theta = \frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}} = \frac{3}{4}$$

9. (d):



We have,

$x^2 - y^2 + 2y - 1 = 0 \Rightarrow x^2 - (y - 1)^2 = 0$

$\Rightarrow (x + y - 1)(x - y + 1) = 0$

Angle bisector of given pair of lines is

$$\frac{x + y - 1}{\sqrt{2}} = \pm\frac{x - y + 1}{\sqrt{2}}$$

$\Rightarrow x + y - 1 = x - y + 1$ | $x + y - 1 = -x + y - 1$

$\Rightarrow 2(y - 1) = 0 \Rightarrow y = 1$ | $\Rightarrow 2x = 0 \Rightarrow x = 0$

Area $= a = \dfrac{1}{2} \times 1 \times 1 = \dfrac{1}{2}$

Orthocentre (0, 1) $\Rightarrow b = 0, c = 1$

Then $4a + b + c = 2 + 0 + 1 = 3$

10. (a): Given, $z^5 = 1$

Then $\dfrac{z^5}{z^5 + z} + \dfrac{z}{1+z} + \dfrac{z^2}{z^2 + 1} + \dfrac{z^5}{z^2 + z^5}$

$$= \frac{1}{1+z} + \frac{z}{1+z} + \frac{z^2}{z^2+1} + \frac{1}{z^2+1} = 1 + 1 = 2$$

11. (d): Equation of tangent is

$y = 3x \pm \sqrt{9a^2 + b^2}$,

Rvp

This is normal to the circle $x^2 + y^2 + 4x + 1 = 0$

This tangent passes through $(-2, 0)$

$$\Rightarrow \quad 0 = -6 \pm \sqrt{9a^2 + b^2} \quad \Rightarrow \quad 6 = \pm\sqrt{9a^2 + b^2}$$

$\Rightarrow \quad 36 = 9a^2 + b^2$

AM $\geq$ GM

$$\Rightarrow \quad \frac{9a^2 + b^2}{2} \geq \sqrt{9a^2 \times b^2}$$

$\Rightarrow \quad 18 \geq 3ab \quad \Rightarrow \quad 6 \geq ab$

12. (c) : We have, $F(x) = \int_0^x f(t)dt$

$$\therefore \quad F(x^2) = \int_0^{x^2} f(t) \cdot dt = x^2(1 + x)$$

Differentiating both sides w.r.t. x, we get

$$f(x^2) \cdot 2x - f(0) \cdot 0 = 2x + 3x^2$$

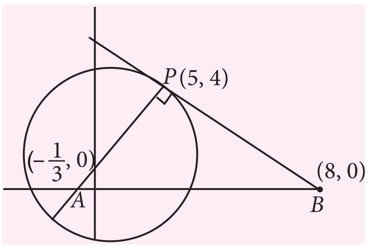
$$\Rightarrow \quad f(x^2) = 1 + \frac{3}{2}x$$

At $x = 2$, $f(4) = 1 + \frac{3}{2} \times 2 = 1 + 3 = 4$

$\therefore$ Unit place of $(985697)^{16}$

$= \text{unit place of } [(985697)^4]^4 = 1^4 = 1$

13. (a, c) :



$$\frac{dy}{dx} = \frac{1-x}{y-1}$$

$$\frac{(y-1)^2}{2} = \frac{-(x-1)^2}{2} + C$$

$\because$ It passes through $(5, 4)$

$\therefore \quad (x - 1)^2 + (y - 1)^2 = 25$

$\Rightarrow \quad x^2 + y^2 - 2x - 2y - 23 = 0$

Equation of tangent at $(5, 4)$ is

$5x + 4y - (x + 5) - (y + 4) - 23 = 0$

$\Rightarrow \quad 4x + 3y - 32 = 0 \qquad \text{...(i)}$

and equation of normal at $(5, 4)$ is $y - 4 = \frac{4-1}{5-1}(x - 5)$

$$\Rightarrow \quad y - 4 = \frac{3}{4}(x - 5)$$

$\Rightarrow \quad 3x - 4y + 1 = 0 \qquad \text{... (ii)}$

$$\therefore \quad \text{Area of } \Delta ABP = \frac{1}{2} \times 4 \times \left(8 + \frac{1}{3}\right) = \frac{50}{3} \text{ sq. units}$$

$\therefore$ Length of perpendicular from $(0, 0)$ to (i) is

$$\left|\frac{0 + 0 - 32}{5}\right| = \frac{32}{5} \text{ units}$$

14. (c) : $\int_0^1 f(x)dx = 0, \int_0^1 xf(x)\,dx = 0, \ldots\ldots, \int_0^1 x^{n-1} f(x)dx = 0$

$$\Rightarrow \quad \int_0^1 \left(x - \frac{1}{2}\right)^n f(x)\,dx = 1$$

15. (a, d) : $\lim_{x \to 5^+} \frac{(x-5)^p}{\ln \cos^q(x-5)} = -1$

$$\Rightarrow \quad \lim_{h \to 0} \frac{h^p}{q \ln \cos h} = -1 \Rightarrow \lim_{h \to 0} \frac{ph^{p-1} \cos h}{-q \sin h} = -1$$

$\Rightarrow \quad p - 2 = 0 \quad i.e. \quad p = 2$

$$\therefore \quad \frac{2}{q} = 1 \quad \Rightarrow \quad q = 2$$

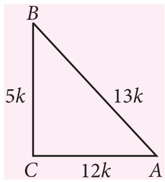
16. (a, b, c) : $313a^2 + 25b^2 + 25c^2 = 120ab + 130ac$

$\Rightarrow \quad 144a^2 - 120ab + 25b^2 + 169a^2 - 130ac + 25c^2 = 0$

$\Rightarrow \quad (12a - 5b)^2 + (13a - 5c)^2 = 0$

$\Rightarrow \quad 12a = 5b \text{ and } 13a = 5c$

$$\Rightarrow \quad \frac{a}{5} = \frac{b}{12} = \frac{c}{13}$$



17. (a, b, c, d)

18. (a, b) : $\{f'(x)\}^2 - 4f(x) \cdot f'(x) + \{f(x)\}^2 = 0$

$$\Rightarrow \quad f'(x) = \frac{4f(x) \pm \sqrt{16(f(x))^2 - 4(f(x))^2}}{2}$$

$$\therefore \quad \frac{f'(x)}{f(x)} = 2 \pm \sqrt{3}$$

$$\therefore \quad \ln f(x) = (2 \pm \sqrt{3})x + c$$

$$\Rightarrow \quad f(x) = ke^{(2 \pm \sqrt{3})x}$$

19. (d): Consider

$$\left(\frac{f(x)}{x^2}\right)' = \frac{x^2 f'(x) - f(x)2x}{x^4} = \frac{xf'(x) - 2f(x)}{x^3}$$

By wrong calculations.

$$\left(\frac{f(x)}{x^2}\right)' = \frac{f'(x)}{2x}$$

$$\therefore \quad \frac{f'(x)}{2x} = \frac{xf'(x) - 2f(x)}{x^3}$$

i.e. $x^2 f'(x) = 2xf'(x) - 4f(x)$

i.e. $4f(x) = (2x - x^2)f'(x)$

i.e. $\dfrac{4}{2x-x^2}=\dfrac{f'(x)}{f(x)}$

$\therefore\ \dfrac{f'(x)}{f(x)}=2\left(\dfrac{1}{x}+\dfrac{1}{2-x}\right)$

On integrating, we get

$\therefore\ \ln f(x) = 2[\ln x - \ln(2-x)] + \ln c$

$\Rightarrow\ f(x)=c\cdot\dfrac{x^2}{(2-x)^2} \quad \therefore\ \lim\limits_{x\to\infty} f(x)=c=4$

$\therefore\ f(x)=\dfrac{4x^2}{(2-x)^2}$

Now, $\lim\limits_{x\to\infty}\left(\dfrac{f(x)}{4}\right)^{\frac{x^2}{2x+1}}=\lim\limits_{x\to\infty}\left(\dfrac{x^2}{(x-2)^2}\right)^{\frac{x^2}{2x+1}}=e^2$

20. (c) : $\sum\limits_{r=1}^{n+2}\dfrac{r^2}{f(r)}=\sum\limits_{r=1}^{n+2}\dfrac{r^2(r-2)^2}{4r^2}=\dfrac{1}{4}\sum\limits_{r=1}^{n+2}(r-2)^2$

$=\dfrac{1}{4}\left(1+\sum\limits_{r=3}^{n+2}(r-2)^2\right)=\dfrac{1}{4}\left(1+\dfrac{n(n+1)(2n+1)}{6}\right)$

21. (b) : Points of intersection of $y=x^2$ and $y=\dfrac{4x^2}{(x-2)^2}$ are given by $\dfrac{x^2}{1}=\dfrac{4x^2}{(x-2)^2}$

i.e. $x^2 - 4x = 0$ *i.e.* $x = 0, x = 4$

$\therefore$ Points of intersection are (0, 0) and (4, 16)

Slope of the common chord = 4

$g'(x) = 2x = 4 \Rightarrow x = 2$

$\therefore$ Required point is (2, 4).

22. (8) : Equation of normal at $\left(\alpha, \dfrac{9}{\alpha}\right)$ to $xy = 9$ is

$$y-\frac{9}{\alpha}=\frac{\alpha^2}{9}(x-\alpha) \qquad \text{... (i)}$$

To have a shortest distance it must pass through (0, 0). $\therefore\ \alpha = 3$

$\therefore$ Point on hyperbola is (3, 3).

Similarly, point on circle is (1, 1).

$\therefore\ S.D.=\sqrt{(3-1)^2+(3-1)^2}=\sqrt{8}$

Required value = 8

23. (0) : $f(2x^2 - 1) = (x^3 + x) f(x)$ ... (i)

Replacing x by $-x$

$f(2x^2 - 1) = -(x^3 + x) f(-x)$ ... (ii)

From (i) and (ii), we get $f(-x) = -f(x)$

Hence $f(x)$ is an odd function and as it is continuous

$\Rightarrow f(0) = 0$

Now, $\lim\limits_{x\to0}\dfrac{f(2x^2-1)}{x}=\lim\limits_{x\to0}(x^2+1)f(x)$

$\lim\limits_{x\to0} f(x)=0 \qquad (\because f(x)$ is continuous at $x = 0)$

$\Rightarrow\ \lim\limits_{x\to0}\dfrac{f(2x^2-1)}{x}=0$

Let $x=\sin\dfrac{\theta}{2}$

$\therefore\ \lim\limits_{\theta\to0}\dfrac{f\left(2\sin^2\frac{\theta}{2}-1\right)}{\sin\frac{\theta}{2}}=0\Rightarrow\lim\limits_{\theta\to0}\dfrac{f(\cos\theta)}{\sin\frac{\theta}{2}}=0$ ... (iii)

$\Rightarrow\ \lim\limits_{x\to0}\dfrac{f(\cos x)}{2\sin\frac{x}{2}\cos\frac{x}{2}}=0\ \Rightarrow\ \lim\limits_{x\to0}\dfrac{f(\cos x)}{\sin x}=0$

24. (2) : Required number of ways

= total – when one row is empty (either 1st or 3rd)

$= {}^{10}C_7 \cdot 7! - {}^2C_1 \cdot 7!$

$=7!\left(\dfrac{10!}{7!\,3!}-2\right)=118\cdot7!$

$\Rightarrow k = 2$

25. (2) : $1920 = 5 \times 3 \times 2^7$

$1920 = 5 \times 3 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

$\therefore$ Number of ways $=\dfrac{9!}{7!}=9\times8$

$1920 = 5 \times 3 \times 4 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1$

$\therefore$ Number of ways $=\dfrac{9!}{5!}=9\times8\times7\times6$

$1920 = 3 \times 5 \times 4 \times 4 \times 2 \times 2 \times 2 \times 1 \times 1$

$\therefore$ Number of ways $=\dfrac{9!}{2!\,2!\,3!}$

$1920 = 3 \times 5 \times 4 \times 4 \times 4 \times 2 \times 1 \times 1 \times 1$

$\therefore$ Number of ways $=\dfrac{9!}{3!\,3!}$

So, total number of ways $=\dfrac{9!}{7!}+\dfrac{9!}{5!}+\dfrac{9!}{2!\,2!\,3!}+\dfrac{9!}{3!\,3!}$

$=\dfrac{9!}{7!}[1+42+210+140]=393\cdot{}^9P_2$

JEE WORK OUTS

PAPER-1

SECTION-I

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct. [Correct ans. 3 marks & wrong ans., no negative mark]

1. A curve that passing through (2, 4) and having subnormal of constant length of 8 units can be
(a) $y^2 = 16x - 16$ (b) $y^2 = -16x + 48$
(c) $x^2 = 16y - 60$ (d) $x^2 = -16y + 68$

2. If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$, then
(a) $x = \frac{1}{5}$ (b) $x + iy = \frac{1}{5}(3 + i)$
(c) $x - iy = \frac{1}{5}(3 + i)$ (d) $x + iy = \frac{1 - i}{1 - 2i}$

3. If $a + b + c = 0$, then the roots of the equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ can be
(a) imaginary (b) real and equal
(c) real and unequal (d) none of these

4. The solution of $\left(\frac{dy}{dx}\right)^2 + 2y\cot x\frac{dy}{dx} = y^2$ is
(a) $y - \frac{c}{1 + \cos x} = 0$ (b) $y = \frac{c}{1 - \cos x}$
(c) $x = 2\sin^{-1}\left(\sqrt{\frac{c}{2y}}\right)$
(d) none of these

5. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$ is equal to
(a) $\frac{n(n+1)(n+2)}{6}$ (b) Σn^2
(c) ${}^nC_3$ (d) ${}^{n+2}C_3$

6. If A and B are two matrices such that $AB = BA$, then $\forall\, n \in N$
(a) $A^nB = BA^n$ (b) $(AB)^n = A^nB^n$
(c) $(A + B)^n = {}^nC_0\, A^n + {}^nC_1\, A^{n-1}B + {}^nC_2\, A^{n-2}B^2 + \ldots + {}^nC_n\, B^n$
(d) $A^{2n} - B^{2n} = (A^n - B^n)(A^n + B^n)$

7. If the graph of the function $f(x)$ is symmetrical about two lines $x = a$ and $x = b$ then $f(x)$ must be periodic with period
(a) $\frac{b - a}{2}$ (b) $b - a$
(c) $2(b - a)$ (d) none of these

8. The function $f(x) = \begin{cases} |x - 3|, & x \geq 1 \\ \frac{x^4}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$ is
(a) continuous at $x = 1$
(b) differentiable at $x = 1$
(c) continuous at $x = 3$
(d) differentiable at $x = 3$

9. Let $f(x) = \sin(\pi x) - 4x(1 - x)$, then
(a) $\frac{\sin(\pi x)}{x(1 - x)} \leq 4\ \forall x \in (0, 1)$
(b) $f'\left(\frac{5}{8}\right) + f'\left(\frac{3}{8}\right) = 0$
(c) $f''(x) = 0$ has at least two solutions in (0, 1)
(d) Rolle's theorem can be applied to $f'(x)$ in $\left[\alpha, \frac{1}{2}\right]$ for some $\alpha \in \left(0, \frac{1}{2}\right)$

By : **Vidyalankar Institute**, Pearl Centre, Senapati Bapat Marg, Dadar (W), Mumbai - 28. Tel.: (022) 24306367

10. Let $\int \sqrt{\frac{x}{1-x^3}}\,dx = \frac{2}{3}g(f(x))+c$. Then

(a) $g(x)=\cot^{-1}(x)$ and $f(x)=\sqrt{\frac{1-x^3}{x^3}}$

(b) $g(x)=\tan^{-1}(x)$ and $f(x)=\sqrt{\frac{1-x^3}{x^3}}$

(c) $g(x)=\cot^{-1}(x)$ and $f(x)=\sqrt{\frac{x^3}{1-x^3}}$

(d) $g(x) = \tan^{-1}x$ and $f(x)=\sqrt{\frac{x^3}{1-x^3}}$

SECTION-II

ONE INTEGER VALUE CORRECT TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive). [Correct ans. 3 marks & wrong ans., no negative mark]

11. If α, β, γ are the roots of the equation $x^3 - 9x^2 + 14x + 24 = 0$, then find the numerical value of $|\alpha+\beta+\gamma+\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\beta\gamma|$.

12. The least degree of a polynomial with integer coefficient whose one of the roots may be cos12° is

13. If $\sin^{-1}x+\sin^{-1}y=\pi$ and, if $x=\lambda y$, then the value of $39^{2\lambda}+5^{\lambda}-1525$ must be

14. The number of solution/s of the equation $\int_{-1}^{x}\left(8t^2+\frac{28t}{3}+4\right)dt = \frac{\frac{3}{2}x+1}{\log_{x+1}\sqrt{x+1}}$ is

15. In a ΔABC, if a is the arithmetic mean and b, c are two geometric means between any two positive numbers. Then $\frac{\sin^3 B+\sin^3 C}{\sin A\sin B\sin C}$ is equal to

16. Let 'm' denotes the number of ways in which 4 different balls of green colour and 4 different balls of red colour can be distributed equally among 4 persons if each person has balls of the same colour and 'n' be corresponding figure when all the four persons have balls of different colour. Find $\frac{(m+n)}{132}$.

17. Let $f(x) = [3 + 4\sin x]$ (where $[\cdot]$ denotes the greatest integer function). If sum of all the values of x in $[\pi, 2\pi]$ where $f(x)$ fails to be differentiable, is $\frac{k\pi}{2}$, then the value of $k/8$ is

18. The polynomial $p(x) = 1 - x + x^2 - x^3 + \ldots + x^{16} - x^{17}$ can be written as a polynomial in y where $y = x + 1$, then let coefficient of y^2 be k then $\left[\frac{k}{400}\right]$ (where $[\cdot]$ denote greatest integer function) is

19. Let A be the centre of the circle $x^2+y^2-2x-4y-20=0$. The tangents at the points $B(1, 7)$ and $C(4, -2)$ on the circle meet at the point D. If Δ denotes the area of the quadrilateral $ABCD$, then $\sqrt{\frac{\Delta}{3}}$ is equal to

20. Find the integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number.

PAPER-2

SECTION-I

SINGLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. [Correct ans. 3 marks & wrong ans. –1]

1. If x and y are positive real numbers and m, n are any positive integers and $E=\frac{x^n y^m}{(1+x^{2n})(1+y^{2m})}$ then

(a) $E>\frac{1}{4}$ (b) $E>\frac{1}{2}$

(c) $E\le\frac{1}{4}$ (d) $E<\frac{1}{8}$

2. Four persons are selected at random out of 3 men, 2 women and 4 children. What is the chance that exactly 2 of them are children?

(a) $\frac{9}{21}$ (b) $\frac{10}{23}$

(c) $\frac{11}{24}$ (d) $\frac{10}{21}$

3. For the equation $x^2 - (k + 1)x + (k^2 + k - 8) = 0$ if one root is greater than 2 and other is less than 2, then k can take any value between

(a) $(-2, 1)$ (b) $(1, 3)$

(c) $\left(-\frac{11}{3}, -2\right)$ (d) none of these

4. If z is a complex number satisfying $|z + 1 - i| \le 1$, then the maximum value of $|z|$ is
(a) $\sqrt{2}$ (b) $\sqrt{2}-1$
(c) $\sqrt{2}+1$ (d) 1

5. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is
(a) 1 (b) –1
(c) 2 (d) none of these

6. For a parabola having focus at S, vertex at A such that $SA = l_1$ units and focal chord PQ of length l_2 units is given, then ar(ΔAPQ) is
(a) $l_1 l_2$ sq. units (b) $l_1\sqrt{l_1 l_2}$ sq. units
(c) $l_2\sqrt{l_1 l_2}$ sq. units (d) $\frac{l_2^3}{2l_1}$ sq. units

7. If $\cos^4\theta \sec^2\alpha, \frac{1}{2}$ and $\sin^4\theta \operatorname{cosec}^2\alpha$ are in A.P., then $\cos^8\theta \sec^6\alpha, \frac{1}{2}$ and $\sin^8\theta \operatorname{cosec}^6\alpha$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

8. The point of intersection of the plane $\vec{r}\cdot 3\hat{i}-5\hat{j}+2\hat{k}=6$ with the straight line passing through the origin and perpendicular to the plane $2x - y - z = 4$ is
(a) $(1, -1, -1)$ (b) $(-1, -1, 2)$
(c) $(4, 2, 2)$ (d) $\left(\frac{4}{3}, \frac{-2}{3}, \frac{-2}{3}\right)$

9. The function $f(x)=\log_e(1+x)-\frac{2x}{2+x}$ is increasing on
(a) $(-1, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty)$ (d) none of these

10. If $\lim_{x\to\infty} f(x^2)=a$ (a finite number), then which of the following is/are true?
(a) $\lim_{x\to\infty} x^2 f'(x^2)=0$
(b) $\lim_{x\to\infty} x^2 f'(x^2)=2a$
(c) $\lim_{x\to\infty} x^4 f''(x^2)=0$
(d) both (a) and (c)

SECTION-II

PARAGRAPH TYPE

This section contains 3 paragraph. Based upon each of the paragraphs 2 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct. [Correct ans. 3 marks & wrong ans. –1]

Paragraph for Q. No. 11 & 12

Suppose equation is $f(x) - g(x) = 0$ or $f(x) = g(x) = y$ say, then draw the graphs of $y = f(x)$ and $y = g(x)$. If graphs of $y = f(x)$ and $y = g(x)$ cuts at one, two, three, …, no points, then number of solutions are one, two, three, …, zero respectively.

11. The number of solutions of $\sin x=\frac{|x|}{10}$ is
(a) 4 (b) 6
(c) 8 (d) none of these

12. Total number of solutions of the equation $3x+2\tan x=\frac{5\pi}{2}$ in $x \in [0, 2\pi]$ is equal to
(a) 1 (b) 2 (c) 3 (d) 4

Paragraph for Q. No. 13 & 14

A and B are two matrices of same order 3×3, where

$$A=\begin{pmatrix}1&2&3\\2&3&4\\5&6&8\end{pmatrix} \text{ and } B=\begin{pmatrix}3&2&5\\2&3&8\\7&2&9\end{pmatrix}$$

13. The value of adj(adj$A$) is equal to
(a) $2A$ (b) $4A$
(c) $8A$ (d) none of these

14. The value of $|\text{adj}(\text{adj } A)|$ is equal to
(a) 9 (b) 16 (c) 25 (d) 1

Paragraph for Q. No. 15 & 16

Consider two functions $f(x)=\lim_{n\to\infty}\left(\cos\frac{x}{\sqrt{n}}\right)^n$ and $g(x) = -x^{4b}$, where $b=\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2+1}$

15. The function $f(x)$ is
(a) e^{-x^2} (b) $e^{-\frac{x^2}{2}}$
(c) e^{x^2} (d) $e^{\frac{x^2}{2}}$

16. Number of solutions of $f(x) + g(x) = 0$ is
(a) 0 (b) 3
(c) 4 (d) none of these

SECTION-III

MATCHING LIST TYPE (ONLY ONE OPTION CORRECT)

This section contains 4 questions, each having two matching lists. Choices for the correct combination of elements from List-I and List-II are given as options (a), (b), (c) and (d), out of which one is correct.[Correct ans. 3 marks & wrong ans. –1]

17. Match the following.

	List-I		List-II
(P)	The value of $\sin(\sin^{-1}1)$ is	1	-1
(Q)	The points $(k, 2-2k)$, $(-k+1, 2k)$ and $(-4-k, 6-2k)$ are collinear if k equal to	2	1
(R)	The value of $\frac{1}{3}\left[\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{5\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right)\right]$ is	3	1/2
(S)	If a, b, c are all different from zero, and $\Delta=\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}$ is equal to zero then the value of $a^{-1}+b^{-1}+c^{-1}$ is	4	$\sqrt{2}\pm1$

Codes:

	P	Q	R	S
(a)	4	1, 4	2	3
(b)	2	1	2, 4	3
(c)	1, 3	3	2	4
(d)	2	1, 3	3	1

18. Match the following.

	List-I		List-II
(P)	Let $f(x)=[x-1]+[1-x]$, $[x]$ is the greatest integer function, a is an integer	1	continuous at $x=a$
(Q)	Let f be as in (P) but a is not an integer	2	$\lim_{x\to0} f(x)$ does not exist
(R)	Let $f(x) = \cot x$	3	$f(a) = 0$
(S)	$f(x)=\dfrac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}$	4	$\lim_{x\to\pi/2} f(x)=\log a$

Codes:

	P	Q	R	S
(a)	4	1	2	3
(b)	2	1	4	3
(c)	1	3	2	4
(d)	3	1	2	4

Cut-offs for IITs may rise due to ambiguities in test : Experts

The first level of Joint Entrance Examination (JEE), for admission to Indian Institute of Technology (IIT) and other engineering institutes, held on Sunday was tougher compared to last year, say experts.

Experts said that the cut-off is expected to rise due to ambiguities in certain questions.

The cut-offs for Common Merit List (CML) in 2013, 2014, 2015 were 113,115 and 105 respectively.

"Few questions in the chemistry paper were ambiguous," said Ramesh Batlish, who teaches at a coaching institute.

About 12 lakh students appeared for the examination at over 2,000 centres in 129 cities in India and abroad.

Students said that Paper 1, which had questions from physics, chemistry and mathematics, was from within the syllabus but few were difficult to solve. Paper 2, that tested students in mathematics, aptitude and drawing, was easier.

The exam conducted by Central Board of Secondary Education (CBSE) was written. The online exam is scheduled for April 8 and 9.

"Few questions in physics were tricky and challenging. Compared to last year this section was a shade tougher," said RL Trikha, director FIITJEE. He said the same was the case with chemistry and mathematics.

"Few questions of mathematics were also lengthy," he said.

Disha Sharma, who appeared for the examination, also said the physics section was difficult.

"The chemistry paper was easy. Mathematics section was a little tricky. I had difficulty in solving few physics questions," said Sharma.

Engineering institutes that accept the applicants of JEE (Main) admission, include National Institutes of Technology and Indian Institutes of Information Technology.

A total of 67 such institutes had participated in the joint seat allocation procedure conducted by JoSAA (Joint Seat Allocation Authority) for 2015-2016.

The top 1.5 lakh candidates of the JEE (Main) exam are eligible to take the JEE (Advanced) for admission to IITs and Indian School of Mines, Dhanbad.

JEE (Advanced) is scheduled for May 23.

Courtesy : The Hindustan Times

19. Match the following.

	List-I		List-II
(P)	$\forall\ x \in R$ the value of the expression $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cdot \cos\left(x + \frac{\pi}{3}\right)$ is equal to	1	1/2
(Q)	Exact value of $\cos 40°(1 - 2\sin 10°)$ is	2	–1/3
(R)	The value of $\sum_{k=3}^{\infty} \sin^k\left(\frac{\pi}{6}\right)$, is	3	1/4
(S)	The value of λ for which the lines $x + y + 1 = 0$; $3x + 2\lambda y + 4 = 0$; $x + y - 3\lambda = 0$ are concurrent is/are	4	5/4

Codes:

	P	Q	R	S
(a)	4	3	1	2
(b)	3	4	1	2
(c)	1	3	2	4
(d)	4	1	3	2

20. Match the following.

	List-I		List-II
(P)	Let a function f is defined as $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$. If f satisfy $f(f(x)) = f(x)$, $\forall\ x \in 1, 2, 3, 4$ then number of such function, is	1	9
(Q)	If m and M are the least and greatest value of $f(x) = (\cos^{-1}x)^2 + (\sin^{-1}x)^2$, then $\frac{M}{m}$ has the value equal to	2	10
(R)	Let x and y be two real numbers such that $2\sin x \sin y + 3\cos y + 6\cos x \sin y = 7$. The value of $\tan^2 x + 2\tan^2 y$, is	3	1200
(S)	Number of ways in which 5 different toys can be distributed in 5 children if exactly one child does not get any toy	4	41

Codes:

	P	Q	R	S
(a)	4	2	1	3
(b)	3	4	1	2
(c)	1	3	2	4
(d)	2	4	1	3

ANSWER KEY

PAPER-1

1.	(a, b)	2.	(b, d)	3.	(b, c)	4.	(a, b, c)		
5.	(a, d)			6.	(a, b, c, d)			7.	(c)
8.	(a, b, c)			9.	(a, b, c, d)			10.	(a, d)
11.	(1)	12.	(4)	13.	(1)	14.	(1)	15.	(2)
16.	(6)	17.	(3)	18.	(2)	19.	(5)	20.	(3)

PAPER-2

1.	(c)	2.	(d)	3.	(d)	4.	(c)	5.	(c)
6.	(b)	7.	(a)	8.	(d)	9.	(a)	10.	(d)
11.	(b)	12.	(c)	13.	(d)	14.	(d)	15.	(b)
16.	(d)	17.	(d)	18.	(d)	19.	(d)	20.	(a)

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Solution Sender of Maths Musing

SET-160

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JEE MAIN

SOLVED PAPER 2016

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1. If $f(x)+2f\left(\frac{1}{x}\right)=3x,\ x\neq 0$, and $S=\{x\in R: f(x)=f(-x)\}$; then S

(a) is an empty set
(b) contains exactly one element
(c) contains exactly two elements
(d) contains more than two elements

2. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary is

(a) $\pi/3$ (b) $\pi/6$
(c) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

3. The sum of all real values of x satisfying the equation $(x^2-5x+5)^{x^2+4x-60}=1$ is

(a) 3 (b) –4 (c) 6 (d) 5

4. If $A=\begin{bmatrix}5a & -b\\ 3 & 2\end{bmatrix}$ and A adj $A = AA^T$, then $5a+b$ is equal to

(a) –1 (b) 5 (c) 4 (d) 13

5. The system of linear equations
$x+\lambda y-z=0,\ \lambda x-y-z=0,\ x+y-\lambda z=0$
has a non-trivial solution for

(a) infinitely many values of λ
(b) exactly one value of λ
(c) exactly two values of λ
(d) exactly three values of λ

6. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary, then the position of the word SMALL is

(a) 46^{th} (b) 59^{th}
(c) 52^{nd} (d) 58^{th}

7. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n,\ x\neq 0$ is 28, then the sum of the coefficients of all the terms in this expansion is

(a) 64 (b) 2187 (c) 243 (d) 729

8. If the 2^{nd}, 5^{th} and 9^{th} terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is

(a) 8/5 (b) 4/3 (c) 1 (d) 7/4

9. If the sum of the first ten terms of the series
$\left(1\frac{3}{5}\right)^2+\left(2\frac{2}{5}\right)^2+\left(3\frac{1}{5}\right)^2+4^2+\left(4\frac{4}{5}\right)^2+\ldots$, is $\frac{16}{5}m$, then m is equal to

(a) 102 (b) 101 (c) 100 (d) 99

10. Let $p=\lim_{x\to 0^+}(1+\tan^2\sqrt{x})^{\frac{1}{2x}}$, then $\log p$ is equal to

(a) 2 (b) 1 (c) 1/2 (d) 1/4

11. For $x\in R$, $f(x)=|\log 2-\sin x|$ and $g(x)=f(f(x))$, then

(a) g is not differentiable at $x=0$
(b) $g'(0)=\cos(\log 2)$
(c) $g'(0)=-\cos(\log 2)$
(d) g is differentiable at $x=0$ and $g'(0)=-\sin(\log 2)$

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12. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right)$.

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point

(a) $(0, 0)$ (b) $\left(0, \frac{2\pi}{3}\right)$

(c) $\left(\frac{\pi}{6}, 0\right)$ (d) $\left(\frac{\pi}{4}, 0\right)$

13. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then

(a) $2x = (\pi + 4)r$ (b) $(4 - \pi)x = \pi r$

(c) $x = 2r$ (d) $2x = r$

14. The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$ is equal to

(a) $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$

(b) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$

(c) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$

(d) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + C$

where C is an arbitrary constant.

15. $\lim_{n \to \infty} \left(\frac{(n+1)(n+2)....3n}{n^{2n}}\right)^{1/n}$ is equal to

(a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$

(c) $\frac{9}{e^2}$ (d) $3\log 3 - 2$

16. The area (in sq. units) of the region $\{(x, y) : y^2 \geq 2x \text{ and } x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is

(a) $\pi - \frac{4}{3}$ (b) $\pi - \frac{8}{3}$

(c) $\pi - \frac{4\sqrt{2}}{3}$ (d) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$

17. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation,

$y(1 + xy)dx = xdy$, then $f\left(-\frac{1}{2}\right)$ is equal to

(a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{4}{5}$

18. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus?

(a) $(-3, -9)$ (b) $(-3, -8)$

(c) $\left(\frac{1}{3}, -\frac{8}{3}\right)$ (d) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$

19. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on

(a) a circle

(b) an ellipse which is not a circle

(c) a hyperbola (d) a parabola

20. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at $(-3, 2)$, then the radius of S is

(a) $5\sqrt{2}$ (b) $5\sqrt{3}$

(c) 5 (d) 10

21. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is

(a) $x^2 + y^2 - 4x + 8y + 12 = 0$

(b) $x^2 + y^2 - x + 4y - 12 = 0$

(c) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

(d) $x^2 + y^2 - 4x + 9y + 18 = 0$

22. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is

(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{3}$

23. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along the line $x = y = z$ is

(a) $3\sqrt{10}$ (b) $10\sqrt{3}$

(c) $\frac{10}{\sqrt{3}}$ (d) $\frac{20}{3}$

24. If the line, $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane $lx + my - z = 9$, then $l^2 + m^2$ is equal to
(a) 26 (b) 18 (c) 5 (d) 2

25. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is
(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

26. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true?
(a) $3a^2 - 26a + 55 = 0$
(b) $3a^2 - 32a + 84 = 0$
(c) $3a^2 - 34a + 91 = 0$
(d) $3a^2 - 23a + 44 = 0$

27. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is NOT true?
(a) E_1 and E_2 are independent
(b) E_2 and E_3 are independent
(c) E_1 and E_3 are independent
(d) E_1, E_2 and E_3 are independent

28. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation
$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is
(a) 3 (b) 5 (c) 7 (d) 9

29. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 minutes from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then the time taken (in minutes) by him, from B to reach the pillar is
(a) 6 (b) 10 (c) 20 (d) 5

30. The Boolean expression $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$ is equivalent to
(a) $\sim p \wedge q$ (b) $p \wedge q$
(c) $p \vee q$ (d) $p \vee \sim q$

SOLUTIONS

1. (c): Change x to $1/x$ in the equation

$$f(x)+2f\left(\frac{1}{x}\right)=3x,\ x\neq 0 \text{ to obtain}$$

$$f\left(\frac{1}{x}\right)+2f(x)=\frac{3}{x}$$

Eliminating $f\left(\frac{1}{x}\right)$ between these two equations, we get

$$3f(x)=\frac{6}{x}-3x \quad i.e.\quad f(x)=\frac{2}{x}-x$$

Now to get the elements of S, we solve
$f(x) = f(-x)$

$$\Rightarrow \frac{2}{x}-x=-\frac{2}{x}+x \Rightarrow \frac{2}{x}-x=0$$

$$\Rightarrow x^2=2 \quad \therefore\ x=\pm\sqrt{2}$$

[Rating : Medium]

2. (d) : Let $\alpha=\frac{2+3i\sin\theta}{1-2i\sin\theta}$

$$\Rightarrow \alpha=\frac{(2+3i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$=\frac{(2-6\sin^2\theta)+i(7\sin\theta)}{1+4\sin^2\theta}$$

As α is to be purely imaginary, we have
Re $(\alpha) = 0 \Rightarrow 2 = 6\sin^2\theta$

$$i.e.\ \sin\theta=\pm\frac{1}{\sqrt{3}}$$

[Rating : Easy]

3. (a) : $a^b = 1$ holds iff
1) $a = 1, b \in R$
2) or $b = 0, a > 0$
The first possibility yields

$x^2 - 5x + 5 = 1 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x - 1)(x - 4) = 0$

$\therefore x = 1, 4$

The 2^{nd} possibility yields

$x^2 + 4x - 60 = 0 \Rightarrow (x + 10)(x - 6) = 0$

$\therefore x = -10, 6$

At these values the base is positive.

The sum of all values = 1 + 4 + 6 – 10 = 1

But none of it matches.

Allow the base to be –1. Then

$x^2 - 5x + 5 = -1 \Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x = 2, 3$

At $x = 2$, $x^2 + 4x - 60$ = even

$x = 3$, $x^2 + 4x - 60$ = odd

So, $x = 2$ is selected.

Sum of value of $x = -10 + 6 + 4 + 1 + 2 = 3$.

This is the best answer out of choices.

[Rating : Hard]

4. (b) : We have $AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$= \begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix}$

A (adj A)= AA^T is known, so equating the two expressions,

$$\begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix} = \begin{bmatrix} 10a+3b & 0 \\ 0 & 10a+3b \end{bmatrix}$$

We have, $10a + 3b = 13$ and $15a - 2b = 0$

On solving, we get $a = 2/5$, $b = 3$

Then, $5a + b = 2 + 3 = 5$

[Rating : Medium]

5. (d) : The system $AX = 0$ has non-trivial solution iff det$A = 0$

i.e., $\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda + 1) - \lambda(-\lambda^2 + 1) - (\lambda + 1) = 0$

$\Rightarrow \lambda^3 - \lambda = 0 \Rightarrow \lambda(\lambda^2 - 1) = 0$

$\therefore \lambda = 0, 1, -1$

[Rating : Medium]

6. (d) : The number of words in all formed by using the letters of the word SMALL = $\dfrac{5!}{2!} = 60$

Let's count backwards.

The 59^{th} word is SMLAL

$\therefore$ 58^{th} word is SMALL.

[Rating : Medium]

7. (d) : The number of terms in the expansion of $\left(1 - \dfrac{2}{x} + \dfrac{4}{x^2}\right)^n$ is ${}^{n+2}C_2$

We have ${}^{n+2}C_2 = 28$ giving $(n + 1)(n + 2) = 56$

Then $n = 6$

$\therefore$ Sum of coefficients $= (1 - 2 + 4)^6 = 3^6 = 729$

[Rating : Medium]

8. (b) : Let d be the common difference and a the first term of the A.P. , then we have

$(a + 4d)^2 = (a + d)(a + 8d)$

Now, $\dfrac{a+4d}{a+d} = \dfrac{a+8d}{a+4d} = \dfrac{4d}{3d} = \dfrac{4}{3}$ and $\neq 0$

[Using properties of ratios.]

[Rating : Easy]

9. (b) : Let us denote the expression as A

$\therefore A = \left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + \ldots$ upto 10 terms

$= \dfrac{8^2}{5^2} + \dfrac{12^2}{5^2} + \ldots.$ upto 10 terms

$= \dfrac{4^2}{5^2}(2^2 + 3^2 + 4^2 \ldots.$ upto 10 terms)

$= \dfrac{16}{25}\left(\dfrac{11 \cdot 12 \cdot 23}{6} - 1\right) = \dfrac{16}{25} \cdot 505 = \dfrac{16}{5} \cdot 101$

As the sum is given to be $\dfrac{16}{5}m$ $\therefore m = 101$

[Rating : Easy]

10. (c): We have, $p = \lim\limits_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{\frac{1}{2x}}$

It is of the form 1^∞ hence the limit is given by

$p = e^{\lim\limits_{x\to 0} \frac{\tan^2\sqrt{x}}{2x}} = e^{\lim\limits_{x \to 0} \frac{1}{2}\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2}$

$= e^{\frac{1}{2}} = \sqrt{e}$ $\quad \therefore \log p = \dfrac{1}{2}$

[Rating : Medium]

11. (b) : As we are concerned about differentiability at '0' in the vicinity of sin$x$

$f(x) = \log 2 - \sin x$

$g(x) = f(f(x)) = \log 2 - \sin(\log 2 - \sin x)$

As g is sum of two differentiable functions, so g is differentiable.

$g'(x) = \cos(\log 2 - \sin x) \cdot \cos x$

Then $g'(0) = \cos(\log 2)$.

[Rating : Hard]

12. (b) : $f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$

$$=\tan^{-1}\left(\frac{1+\sin x}{\sqrt{1-\sin^2 x}}\right)=\tan^{-1}\left(\frac{1+\sin x}{\sqrt{\cos^2 x}}\right)$$

$$=\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)\quad \text{As } x\in\left(0,\ \frac{\pi}{2}\right)$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\frac{\pi}{4}+\frac{x}{2}$$

$\therefore\ f'(x)=\frac{1}{2}\qquad \therefore\ f'\left(\frac{\pi}{6}\right)=\frac{1}{2}$

Equation of normal is $y-\frac{\pi}{3}=-2\left(x-\frac{\pi}{6}\right)$

i.e. $2x+y=\frac{2\pi}{3}$

It passes through $\left(0,\ \frac{2\pi}{3}\right)$

[Rating : Hard]

13. (c): We have from hypothesis, $4x+2\pi r=2$

$\therefore\ r=\frac{1-2x}{\pi}$

Area, $A=x^2+\pi r^2=x^2+\frac{\pi}{\pi^2}(2x-1)^2=x^2+\frac{1}{\pi}(2x-1)^2$

For maximum/minimum

$\frac{dA}{dx}=0\ \Rightarrow\ 2x+\frac{4}{\pi}(2x-1)=0$

$\therefore\ x=\frac{2}{\pi+4}$

Also, $\frac{d^2A}{dx^2}>0$ at this value. Thus there is a minimum.

Again, $r=\frac{1}{\pi+4}$, On comparing, $x=2r$

[Rating : Hard]

14. (b) : Let $I=\int\frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$

$$=\int\frac{\frac{2x^{12}+5x^9}{x^{15}}}{\left(\frac{x^5+x^3+1}{x^5}\right)^3}dx=\int\frac{\frac{2}{x^3}+\frac{5}{x^6}}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}dx$$

Put, $1+\frac{1}{x^2}+\frac{1}{x^5}=u$, so that $\frac{du}{dx}=-\left(\frac{2}{x^3}+\frac{5}{x^6}\right)$

The integral reduces to

$$I=-\int\frac{du}{u^3}=\frac{1}{2u^2}+C=\frac{x^{10}}{2(x^5+x^3+1)^2}+C$$

[Rating : Hard]

15. (b) : From the definition of limit as sum

$$\lim_{n\to\infty}\left(\frac{(n+1)(n+2)....3n}{n^{2n}}\right)^{1/n}=e^{\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{2n}\ln\left(1+\frac{r}{n}\right)}$$

$$=e^{\int_0^2\ln(1+x)dx}$$

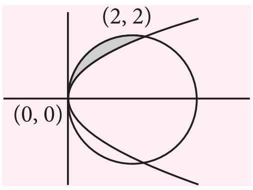
Now, $\int_0^2\ln(1+x)dx=[(x+1)\ln(x+1)-x]_0^2$

$=3\ln 3-2=\ln 27-2$

$\therefore$ Required limit $=e^{\ln 27-2}=\frac{27}{e^2}$

[Rating : Medium]

16. (b) : The area of the required region is shaded.



$\therefore$ Area $=\frac{\pi\cdot 2^2}{4}-\sqrt{2}\int_0^2\sqrt{x}dx$

$$=\pi-\sqrt{2}\cdot\left.\frac{x^{3/2}}{3/2}\right|_0^2=\pi-\sqrt{2}\cdot\frac{2}{3}\cdot 2\sqrt{2}\ =\pi-\frac{8}{3}$$

[Rating : Easy]

17. (d) : The differential equation can be rewritten as

$xdy=ydx+xy^2dx$

$\Rightarrow\ \frac{xdy-ydx}{y^2}=xdx$

On integrating, we get, $-\frac{x}{y}=\frac{x^2}{2}+C$

As the curve passes through $(1,-1)$, we have

$1=\frac{1}{2}+C\quad\therefore\ C=\frac{1}{2}$

Now the curve $f(x)=x^2+1+\frac{2x}{y}=0$

$\Rightarrow\ y=-\frac{2x}{1+x^2}\quad\therefore\ f\left(-\frac{1}{2}\right)=\frac{-2(-1/2)}{1+\frac{1}{4}}=\frac{4}{5}$

[Rating : Medium]

18. (c): Coordinates of $A\equiv(1,2)$

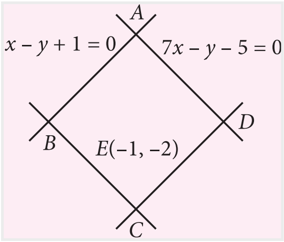
$\therefore$ Slope of $AE=2$

$\Rightarrow$ Slope of $BD = -\frac{1}{2}$

$\Rightarrow$ Eq. of BD is $\frac{y+2}{x+1} = -\frac{1}{2}$

$\Rightarrow$ $x + 2y + 5 = 0$

$\therefore$ Co-ordinates of $D = \left(\frac{1}{3}, \frac{-8}{3}\right)$



[Rating : Hard]

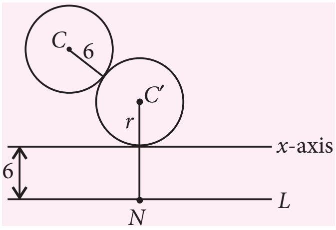
19. (d) : The equation of circle is

$x^2 + y^2 - 8x - 8y - 4 = 0$

$\Rightarrow$ $(x - 4)^2 + (y - 4)^2 = 36 = 6^2$

$\therefore$ Radius = 6.

Consider a line 6 units below the x-axis.



We have $C'C = C'N = r + 6$

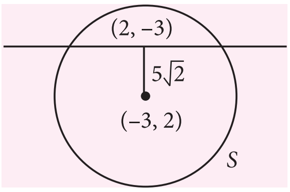
Thus the locus of C' is a parabola with C as focus and L as directrix.

[Rating : Hard]

20. (b) : The circle is $x^2 + y^2 - 4x + 6y - 12 = 0$

$(x - 2)^2 + (y + 3)^2 = 25 = 5^2$

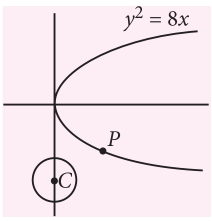
Now,



Radius of $S = \sqrt{25+50} = \sqrt{75} = 5\sqrt{3}$

[Rating : Medium]

21. (a) : The geometry of the situation is as follows.



The point P must lie on the normal common to circle and parabola.

Let the normal be in parametric form.

$y + tx = 4t + 2t^3$

As it has to pass through $(0, -6)$, we have

$t^3 + 2t + 3 = 0$ gives $(t + 1)(t^2 - t + 3) = 0$

The only real value is $t = -1$.

So point P becomes $P(2, -4)$. We have $CP = 2\sqrt{2}$

The equation of circle is $(x - 2)^2 + (y + 4)^2 = (2\sqrt{2})^2 = 8$

i.e. $x^2 + y^2 - 4x + 8y + 12 = 0$

[Rating : Hard]

22. (c): We have $2b = ae$ and $\frac{2b^2}{a} = 8$

Also, we have $b^2 = a^2(e^2 - 1)$

Now eliminating a and b from these equations

$$\frac{e^2}{4} = e^2 - 1 \Rightarrow 4 = 3e^2$$

$$\therefore \quad e = \frac{2}{\sqrt{3}} \quad \text{as } e > 0$$

[Rating : Easy]

23. (b) : The equation of line parallel to $x = y = z$ and passing through $(1, -5, 9)$ is

$$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = k \text{ (say)}$$

Let $A(k + 1, k - 5, k + 9)$ be the point of intersection of line and plane.

We have, $k + 1 - k + 5 + k + 9 = 5 \Rightarrow k = -10$

$\therefore$ The point is $(-9, -15, -1)$

Required distance $= \sqrt{(1+9)^2 + (-5+15)^2 + (9+1)^2}$

$= 10\sqrt{3}$

[Rating : Medium]

24. (d) : As the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane $lx + my - z = 9$, we have

$3l - 2m + 4 = 9$. Also, $2l - m - 3 = 0$

Solving for l and m we get $l = 1$, $m = -1$

So, $l^2 + m^2 = 2$

[Rating : Easy]

OPINION POLL

Do you agree with the govt's decision to not give weightage to Class XII board marks in JEE (Main)?

A	B	C
Yes	No	Can't say
67%	29%	4%

25. (d) : We have, $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$

$\Rightarrow\ (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$

On comparing, $\vec{a}\cdot\vec{b}=-\frac{\sqrt{3}}{2}$

Then $\theta=\frac{5\pi}{6}$

[Rating : Medium]

26. (b) : By formula, S.D. $=\sqrt{\frac{\sum x_i^2}{n}-\left(\frac{\sum x_i}{n}\right)^2}$

Now, $\frac{49}{4}=\frac{4+9+a^2+121}{4}-\left(\frac{16+a}{4}\right)^2$

$\Rightarrow\ 49\cdot 4 = 4(134+a^2)-(256+32a+a^2)$

$\Rightarrow\ 3a^2-32a+84=0$

[Rating : Hard]

27. (d) : $P(E_1)=\frac{6}{6\cdot 6}=\frac{1}{6}$

$P(E_2)=\frac{6}{6\cdot 6}=\frac{1}{6}$; $P(E_3)=\frac{3\cdot 3\cdot 2}{6\cdot 6}=\frac{1}{2}$

(For sum to be odd combination as odd + even or even + odd)

$P(E_1\cap E_2)=\frac{1}{6\cdot 6}=P(E_1)\cdot P(E_2)$

$P(E_1\cap E_3)=\frac{1\cdot 3}{6\cdot 6}=P(E_1)\cdot P(E_3)$

$P(E_2\cap E_3)=\frac{1\cdot 3}{6\cdot 6}=P(E_2)\cdot P(E_3)$

As $P(X\cap Y)=P(X)\cdot P(Y)$ the event X and Y are independent.

Also, $P(E_1\cap E_2\cap E_3)=0$ as the event cannot happen.

So, E_1, E_2, E_3 are pairwise independent, but they together are not independent.

[Rating : Hard]

28. (c): $\cos x+\cos 2x+\cos 3x+\cos 4x=0$

Using sum – product formula, we have

$(\cos x+\cos 3x)+(\cos 2x+\cos 4x)=0$

$\Rightarrow\ 2\cos x\cos 2x+2\cos x\cos 3x=0$

$\Rightarrow\ 2\cos x(\cos 2x+\cos 3x)=0$

$\Rightarrow\ 2\cos x\cdot 2\cos\frac{5x}{2}\cos\frac{x}{2}=0$

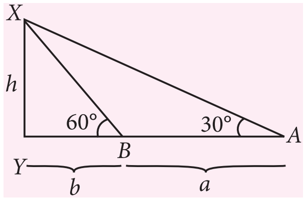
In $x\in[0, 2\pi)$ we have the solution as

$x=\frac{\pi}{2},\ \frac{3\pi}{2},\ \pi,\ \frac{\pi}{5},\ \frac{3\pi}{5},\ \frac{7\pi}{5},\ \frac{9\pi}{5}$

Thus we have 7 solutions.

[Rating : Easy]

29. (d) :



We have $\tan 30^\circ=\frac{h}{a+b}$ and $\tan 60^\circ=\frac{h}{b}$

Eliminating h, we have

$\frac{\sqrt{3}}{1/\sqrt{3}}=\frac{a+b}{b}\ \Rightarrow\ a+b=3b$

$\therefore\ a=2b$

As the man covers distance a in 10 minutes, he will take 5 minutes to reach the pillar, as he has to travel half the distance.

[Rating : Easy]

30. (c): We have $(p\wedge\sim q)\vee q\vee(\sim p\wedge q)$

$\equiv((p\vee q)\wedge(\sim q\vee q))\vee(\sim p\wedge q)$

$\equiv(p\vee q)\wedge(t)\vee(\sim p\wedge q)$

$\equiv(p\vee q)\vee(\sim p\wedge q)\equiv p\vee q$

[Rating : Medium]

The notion that girls are not good with numbers and science is just a myth, if data from a nationwide survey of more than 2.7 lakh students is any indicator. The survey conducted on Class X student showed girls performed on an equal footing with boys in mathematics, science and social sciences.

The study, however, upheld another common conception — that girls have better language skills. Girls outperformed boys in English and other languages in the survey conducted in 2015 by the National Council of Educational Research and Training (NCERT) in 7,216 schools following different boards across 33 states and Union territories.

The study also highlighted rampant under-performance among students in rural settings, those studying in government schools and hailing from underprivileged backgrounds, such as Dalits and tribals.

Another disturbing trend was the poor showing in science and maths by students in a majority of states. Scores in science were below the national average in 24 states.

JEE Advanced

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PRACTICE PAPER 2016

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TIO I

Single Correct Answer Type

1. Let $f(x) = \sin\theta(x^2 - 2)((\sin\theta)x + \cos\theta)$, $(\theta \neq m\pi, m \in I)$. Then $f(x)$ has

(a) local maxima at certain $x \in R^+$

(b) a local maxima at certain $x \in R^-$

(c) a local minima at certain $x = 0$

(d) a local minima at certain $x \in R^-$

2. If $\sin^{-1}\left(\sin\dfrac{33\pi}{7}\right)+\cos^{-1}\left(\cos\dfrac{46\pi}{7}\right)$ $+\tan^{-1}\left(-\tan\dfrac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\dfrac{19\pi}{8}\right)\right)=\dfrac{a\pi}{b}$ where a and b are in their lowest form, then $(a + b)$ equal to

(a) 17 (b) 20

(c) 23 (d) none of these

3. If a line with direction ratios 2 : 2 : 1 intersects the line $\dfrac{x-7}{3}=\dfrac{y-5}{2}=\dfrac{z-3}{1}$ and $\dfrac{x-1}{2}=\dfrac{y+1}{4}=\dfrac{z+1}{3}$ at A and B then $AB =$

(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{3}$ (d) 3

4. Least value modulus of slope of a line for which the line may touch the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{(a^3+a^2+a)^2}=1$ is (a is non zero real)

(a) $\sqrt{3}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{3}{4}$

5. Length of latus rectum of ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{b^2}=1$, if the normal at an end of latus rectum passes through one extremity of the minor axis, is

(a) $2-\sqrt{3}$ (b) $6-\sqrt{3}$

(c) $6-2\sqrt{3}$ (d) $6-2\sqrt{5}$

6. Radius of circle which is drawn on a normal chord of $y^2 = 4x$ as diameter and it passes through vertex is

(a) $\sqrt{3}$ (b) $\dfrac{1}{\sqrt{3}}$

(c) $2\sqrt{3}$ (d) $3\sqrt{3}$

7. There is a point P on the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$ such that its distance to the right directrix is the average of its distance to the two foci. Let the x-coordinate of P be m/n with m and n being integers, $(n > 0)$ having no common factor except 1. Then $n - m$ equals

(a) 59 (b) 69 (c) –59 (d) –69

8. Suppose two complex numbers $z = a + ib$; $w = c + id$ satisfy the equation $\dfrac{z+w}{z}=\dfrac{w}{z+w}$ then

(a) both a and c are zeros

(b) both b and d are zeros

(c) both b and d must be non-zeroes

(d) at least one of b and d is non-zero

9. The value of the expression

$$\sum_{0\le i<j\le n}\sum (-1)^{i+j-1}\ {}^nC_i\cdot{}^nC_j \text{ is}$$

(a) ${}^{2n-1}C_n$ (b) ${}^{2n}C_n$

(c) ${}^{2n+1}C_n$ (d) none of these

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10. Let $P(x)$ be a polynomial with degree 2009 and leading co-efficient unity such that $P(0) = 2008$, $P(1) = 2007$, $P(2) = 2006, \ldots, P(2008) = 0$ then the value of $P(2009) = (n!) - a$, where n and a are natural numbers then value of $(n + a)$ is
(a) 2010 (b) 2009 (c) 2011 (d) 2008

11. Let $f(x) = x^2 + \lambda x + \mu \cos x$, λ being an integer and μ a real number. The number of ordered pairs (λ, μ) for which the equations $f(x) = 0$ and $f(f(x)) = 0$ have the same (non empty) set of real roots is
(a) 4 (b) 6
(c) 8 (d) infinite

12. Mr. A makes a bet with Mr. B that in a single throw with two dice he will throw a total of seven before B throws four. Each of them has a pair of dice and they throw simultaneously until one of them wins, equal throws being disregarded. Probability that B wins, is
(a) 1/3 (b) 4/11 (c) 5/16 (d) 6/17

TIO II

Multiple Correct Answer Type

13. Consider the planes $P_1 : 2x + y + z + 4 = 0$
$P_2 : y - z + 4 = 0$ and $P_3 : 3x + 2y + z + 8 = 0$
Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, and P_1 and P_2 respectively. Then,
(a) At least two of the lines L_1, L_2 and L_3 are non-parallel
(b) At least two of the lines L_1, L_2 and L_3 are parallel
(c) The three planes intersect in a line
(d) The three planes form a triangular prism

14. If two tangents can be drawn to the different branches of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ from the point (α, α^2), then
(a) $\alpha \in (-2, 0)$ (b) $\alpha \in (0, 2)$
(c) $\alpha \in (-\infty, -2)$ (d) $\alpha \in (2, \infty)$

15. The points on axis of parabola $3x^2 + 4x - 6y + 8 = 0$ from which three distinct normals can be drawn to it are
(a) $\left(\frac{-2}{3}, 2\right)$ (b) $\left(\frac{-2}{3}, 3\right)$
(c) $\left(\frac{-2}{3}, 4\right)$ (d) $\left(\frac{-2}{3}, 1\right)$

16. If three numbers are chosen randomly from the set $\{1, 3, 3^2, \ldots, 3^n\}$ without replacement, then the probability that they form an increasing geometric progression is
(a) $\frac{3}{2n}$ if n is odd
(b) $\frac{3}{2n}$ if n is even
(c) $\frac{3n}{2(n^2-1)}$ if n is even
(d) $\frac{3n}{2(n^2-1)}$ if n is odd

17. The number $\frac{5^k + 3}{4} (k \in N)$, when divided by 10, may leave remainder
(a) 2 (b) 6 (c) 7 (d) 8

18. The values of a for which $\frac{x^3 - 6x^2 + 11x - 6}{x^3 + x^2 - 10x + 8} + \frac{a}{30} = 0$ does not have a real solution is
(a) −10 (b) 12 (c) 5 (d) −30

19. If $A(z_1)$, $B(z_2)$, $C(z_3)$ are three points in argand plane where $|z_1 + z_2| = |z_1| - |z_2|$ and $|(1 - i)z_1 + iz_3| = |z_1| + |z_3 - z_1|$, then
(a) A, B and C lie on a fixed circle with centre $\left(\frac{z_2 + z_3}{2}\right)$
(b) A, B, C form right angle triangle
(c) A, B, C form an equilateral triangle
(d) A, B, C form an obtuse angle triangle

20. If E and F are two independent events, such that $P(E \cap F) = \frac{1}{6}$, $P(E^C \cap F^C) = \frac{1}{3}$ and $(P(E) - P(F))(1 - P(F)) > 0$, then
(a) $P(E) = \frac{1}{2}$ (b) $P(E) = \frac{1}{4}$
(c) $P(F) = \frac{1}{3}$ (d) $P(F) = \frac{2}{3}$

21. The number of isosceles triangles with integer sides if no side exceeds 2008 is
(a) $(1004)^2$ if equal sides do not exceed 1004
(b) $2(1004)^2$ if equal sides exceed 1004
(c) $3(1004)^2$ if equal sides have any length ≤ 2008
(d) $(2008)^2$ if equal sides have any length ≤ 2008

TIO III

Comprehension Type

Paragraph for Question No. 22 to 23

Consider the planes $S_1 : 2x - y + z = 5$, $S_2 : x + 2y - z = 4$ having normals N_1 and N_2 respectively. $P(2, -1, 0)$ and $Q(1, 1, -1)$are points on S_1 and S_2 respectively.

22. A vector of magnitude $\sqrt{140}$ units and lies along the line of intersection of S_1 and S_2 is

(a) $2(5\hat{i}+3\hat{j}-\hat{k})$ (b) $2(\hat{i}+3\hat{j}+5\hat{k})$
(c) $2\hat{i}-6\hat{j}-10\hat{k}$ (d) $2(3\hat{i}-\hat{j}+5\hat{k})$

23. The distance of the origin from the plane passing through the point (1, 1, 1) and whose normal is perpendicular to N_1 and N_2 is

(a) $\frac{9}{\sqrt{61}}$ (b) $\frac{11}{\sqrt{35}}$ (c) $\frac{10}{\sqrt{61}}$ (d) $\frac{7}{\sqrt{35}}$

Paragraph for Question No. 24 to 26

In a parallelogram $OABC$ with position vectors of A as $3\hat{i}+4\hat{j}$ and C as $4\hat{i}+3\hat{j}$ with reference to O as origin. A point E is taken on the side BC which divides it in the ratio of 2 : 1. Also, the line segment AE intersects the line bisecting the $\angle AOC$ internally at P. CP when extended meets AB at point F.

24. The position vector of P is

(a) $\hat{i}+\hat{j}$ (b) $\frac{2}{3}(\hat{i}+\hat{j})$
(c) $\frac{13}{3}(\hat{i}+\hat{j})$ (d) $\frac{21}{5}(\hat{i}+\hat{j})$

25. The equation of line parallel to $\overline{CP}$ and passing through (2,3,4) is

(a) $\frac{x-2}{1}=\frac{y-3}{5};z=4$ (b) $\frac{x-2}{1}=\frac{y-3}{6};z=4$
(c) $\frac{x-2}{2}=\frac{y-3}{5};z=3$ (d) $\frac{x-2}{3}=\frac{y-3}{5};z=3$

26. The equation of plane containing line AC and at a maximum distance from B is

(a) $\vec{r}\cdot(\hat{i}+\hat{j})=7$ (b) $\vec{r}\cdot(\hat{i}-\hat{j})=7$
(c) $\vec{r}\cdot(2\hat{i}-\hat{j})=7$ (d) $\vec{r}\cdot(3\hat{i}+4\hat{j})=7$

Paragraph for Question No. 27 to 29

Consider the hyperbola $xy + x - y - 9 = 0$ and a line $x = 4$, which intersect the transverse axis at P. A parabola with vertex at P and axis parallel to y-axis passes through (1, 3), then

27. Equation of parabola is

(a) $y = x^2 - 2x + 4$ (b) $9y = x^2 - 8x + 20$
(c) $9y = x^2 - 8x + 34$ (d) $18y = x^2 - 8x + 20$

28. If tangents at P of parabola intersect hyperbola at P', then area of triangle PCP' (C is centre of hyperbola) is

(a) 1/2 sq. unit (b) 1 sq. unit
(c) 1/4 sq. unit (d) 2 sq. units

29. Equation of latus rectum of hyperbola which does not pass through IIIrd quadrant is

(a) $y + x - 2 = 0$ (b) $y + x + 1 = 0$
(c) $y + x + 4 = 0$ (d) $y + x - 8 = 0$

Paragraph for Question No. 30 to 32

Consider the parabolas $C_1 : y = x^2 + 1$ and $C_2 : x = y^2 + 1$. If PQ is the shortest distance and R, S are the points of contact of common tangent then

30. Distance of common tangent from origin is

(a) $\frac{3}{\sqrt{2}}$ (b) $\frac{3}{4}$ (c) $\frac{3}{4\sqrt{2}}$ (d) $\frac{3}{3\sqrt{3}}$

31. Length of PQ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{3}{2\sqrt{2}}$ (d) $\frac{1}{2}$

32. Area of region $PQRS$ is

(a) 1/4 sq. unit (b) 5/4 sq. units
(c) 7/4 sq. units (d) $5/\sqrt{2}$ sq. units

Paragraph for Question No. 33 to 35

$H : x^2 - y^2 = 9$, $P : y^2 = 4(x - 5)$, $L : x = 9$.

33. If L is a chord of contact of the hyperbola H, then the equation of the corresponding pair of tangents

(a) $9x^2 - 8y^2 + 18x - 9 = 0$
(b) $9x^2 - 8y^2 - 18x + 9 = 0$
(c) $9x^2 - 8y^2 - 18x - 9 = 0$
(d) $9x^2 - 8y^2 + 18x + 9 = 0$

34. If R is the point of intersection of the tangents to H at the extremities of the chord L, then equation of the chord of contact of R with respect to the parabola P is

(a) $x = 7$ (b) $x = 9$
(c) $y = 7$ (d) $y = 9$

35. If the chord of contact of R (as in above question) with respect to the parabola P meets the parabola at T and T', S is the focus of the parabola, then area of the triangle STT' is equal to

(a) 8 sq. units (b) 9 sq. units
(c) 12 sq. units (d) 16 sq. units

Paragraph for Question No. 36 to 38

$P(a, 5a)$ and $Q(4a, a)$ are two points. Two circles are drawn through these points touching the axis of y.

36. Centre of these circles are at

(a) $(a, a), (2a, 3a)$ (b) $\left(\frac{205a}{18}, \frac{29a}{3}\right), \left(\frac{5a}{2}, 3a\right)$

(c) $\left(3a, \frac{29a}{3}\right), \left(\frac{205a}{9}, \frac{29a}{18}\right)$

(d) none of these

37. Angle of intersection of these circles is

(a) $\tan^{-1}(4/3)$ (b) $\tan^{-1}(40/9)$

(c) $\tan^{-1}(84/187)$ (d) $\pi/4$

38. If C_1, C_2 are the centres of these circles then area of ΔOC_1C_2, where O is the origin, is

(a) a^2 sq. units (b) $5a^2$ sq. units

(c) $10a^2$ sq. units (d) $20a^2$ sq. units

TIO I

Matrix Match Type

39. Match the following.

	Column I		Column II
(A)	The maximum value of $\sin(\cos x) + \cos(\sin x)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is	(p)	$\cos(\cos 1)$
(B)	The minimum value of $\sin(\cos x) + \cos(\sin x)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is	(q)	$1 + \cos 1$
(C)	The maximum value of $\cos(\cos(\sin x))$ is	(r)	$\cos 1$
(D)	The minimum value of $\cos(\cos(\sin x))$ is	(s)	$1 + \sin 1$

40. In the column-I and column-II differential equations and their corresponding solutions are given. Match them.

	Column I		Column II
(A)	$(\cos x)\frac{dy}{dx} + y\sin x = 1$	(p)	$y = \frac{\sin x + c}{1 + x^2}$
(B)	$x\frac{dy}{dx} + y = x\sin x$	(q)	$y = \sin x + c\cos x$
(C)	$(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$	(r)	$xy = \sin x - x\cos x + c$
(D)	$\frac{dy}{dx} + (2\tan x)y = \sin x$	(s)	$y = \cos x + c\cos^2 x$

41. Match the following.

	Column I		Column II
(A)	The minimum value of ab if roots of the equation $x^3 - ax^2 + bx - 2 = 0$ are positive, is	(p)	24
(B)	The number of divisors of the form $12\lambda + 6(\lambda \in N)$ of the number 25200 are	(q)	3
(C)	The number of quadrilateral formed in an octagon having two adjacent sides common with the polygon are	(r)	12
(D)	The number of solution of the equation $[\cos x] = \tan x$, where $[\cdot]$ denotes greatest integer function $\forall\, x \in [0, 6\pi]$ are	(s)	18

42. Match the following.

	Column I		Column II
(A)	Number of ways to select n objects from $3n$ objects of which n are identical and rest are different is $k^{2k-1} + \frac{1}{k}\frac{(kn)!}{(n!)^2}$, k is	(p)	3
(B)	Number of interior point when diagonals of a convex polygon of n side intersect if no three diagonal pass through the same interior point is ${}^nC_\lambda$, then λ is	(q)	2
(C)	Five digit number of different digit can be made in which digit are in descending order is ${}^{10}C_\mu$ then μ is	(r)	4
(D)	Number of term in expansion of $(1 + 3^{1/3})^6$ which are free from radical sign	(s)	5
		(t)	1

TIO

Integer Answer Type

43. If length of the side BC of a ΔABC is 4 cm and $\angle BAC = 120°$, then the distance between incentre and excentre of the circle touching the side BC internally is

44. In parallelogram $ABCD$, P is any point inside parallelogram such that sum of area of ΔAPD and ΔBPC is 3 sq. units, then area of parallelogram $ABCD$ is

45. $A = (-3, 0)$ and $B = (3, 0)$ are the extremities of the base AB of triangle PAB. If the vertex P varies such that the internal bisector of angle APB of the triangle divides the opposite side AB into two segments AD and BD such that $AD : BD = 2 : 1$, then the maximum value of the length of the altitude of the triangle drawn through the vertex P is

46. Find the maximum value of $(\log_{2^{1/5}} a)\cdot(\log_{2^{1/2}} b)$. It is given that coefficient of 2^{nd}, 3^{rd} and 4^{th} term in expansion of $(a + b)^n$ are in A.P and the value of 3^{rd} term is equal to 84 $(a, b > 1)$.

47. Let $S=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\ldots..$

$$+\sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}},$$

then find $|2000(S - 2000)|$.

48. Consider two polynomials $f(x)$ and $g(x)$ as $g(x)=\sum_{r=0}^{200}\alpha_r x^r$ and $f(x)=\sum_{r=0}^{200}\beta_r x^r$. Given:

(i) $\beta_r = 1\ \forall\ r \ge 100$ (ii) $f(x + 1) = g(x)$.

Let $A=\sum_{r=100}^{200}\alpha_r$. Find the remainder when A is divided by 15.

49. A sequence is obtained by deleting all perfect squares from set of natural numbers. The remainder when the 2003^{rd} term of new sequence is divided by 2048, is

50. If $\alpha=e^{i2\pi/7}$ and $f(x)=A_0+\sum_{k=1}^{20}A_k x^k$ and the value of $f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^6 x)$ is $k(A_0 + A_7x^7 + A_{14}x^{14})$, then find the value of k.

OLUTIO

1. (b) : $f(x) = (\sin^2\theta)x^3 + \frac{1}{2}\sin 2\theta\cdot x^2 - 2\sin^2\theta\cdot x - \sin 2\theta$

$f'(x) = (3\sin^2\theta)x^2 + x\sin 2\theta - 2\sin^2\theta$

Then $D > 0$ and product of roots < 0

So $f(x)$ has local maxima at some $x \in R^-$ and local minima at some $x \in R^+$

2. (b) : On solving, we get $\frac{a\pi}{b}=\frac{13\pi}{7}\Rightarrow 13+7=20$.

3. (d) : $A(7 + 3\alpha, 5 + 2\alpha, 3 + \alpha)$,
$B(1 + 2\beta, -1 + 4\beta, -1 + 3\beta)$

Dr's of AB are $2 : 2 : 1$

$$\frac{6+3\alpha-2\beta}{2}=\frac{3+\alpha-2\beta}{1}=\frac{4+\alpha-3\beta}{1}$$

$\alpha = -2,\ \beta = 1$

$A(1, 1, 1),\quad B(3, 3, 2)$

$\therefore\ AB = 3$ units

4. (d) : Let slope of line is m

(asymptotes are given by $y=\pm\frac{b}{a}x$)

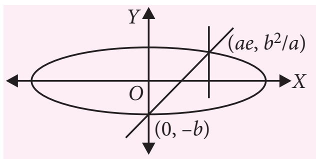
$a^2m^2 - (a^3 + a^2 + a)^2 \ge 0$

$m^2 \ge (a^2 + a + 1)^2$

$m^2\ge\left(\frac{3}{4}\right)^2,\ |m|\ge\frac{3}{4}$

5. (d) : Normal at $(ae, b^2/a)$ and pass through $(0, -b)$

$\Rightarrow\ e^4 + e^2 - 1 = 0$



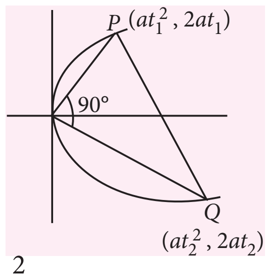
$\Rightarrow\ e^2=\frac{\sqrt{5}-1}{2}$

Also $b^2 = a^2(1 - e^2)$

$$=4\left(1-\frac{\sqrt{5}-1}{2}\right)=2(3-\sqrt{5})$$

Length of L.R. $=\frac{2b^2}{a}=\frac{4(3-\sqrt{5})}{2}=6-2\sqrt{5}$

6. (d) : Normal chord is PQ and normal is drawn at $P(at_1^2, 2at_1)$



$t_2=-t_1-\frac{2}{t_1}$.

Also $t_1t_2 = -4$

$\Rightarrow\ -\frac{4}{t_1}=-t_1-\frac{2}{t_1}$

$\frac{2}{t_1}=t_1\ \Rightarrow\ t_1=\sqrt{2},\ t_2=-\sqrt{2}-\frac{2}{\sqrt{2}}$.

$P\equiv(2,\ 2\sqrt{2}),\quad Q\equiv(8,\ -4\sqrt{2})$

$PQ=\sqrt{36+72}=\sqrt{108}$, $r=\frac{PQ}{2}=\sqrt{27}=3\sqrt{3}$

7. (b) : For the hyperbola, $e=\frac{5}{4}$

From the equation $2\left|x-\frac{a}{e}\right|=|ex+a|+|ex-a|$

It turns out that P has to be on the left branch. x-coordinate is found to be $-64/5$.

8. (c) : $(z + w)^2 = zw\ \Rightarrow\ z^2 + zw + w^2 = 0$

Let $\frac{z}{w}=t\ \Rightarrow\ \frac{z}{w}=\frac{-1\pm\sqrt{3}i}{2}$

$$\arg z - \arg w = \frac{2\pi}{3} \quad \text{or} \quad \arg z - \arg w = -\frac{2\pi}{3}$$

9. (c) : Let the required value be S

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j-1}\ {}^nC_i{}^nC_j = \sum_{i=0}^{n}(-1)^{2i-1}({}^nC_i)^2 + 2S$$

$$= -\sum_{i=0}^{n}({}^nC_i)^2 + 2S$$

$S = {}^{2n-1}C_{n-1} = {}^{2n-1}C_n$

10. (a) : $P(x) - 2008 + x = x(x-1)(x-2)(x-3) \ldots (x-2008)$

Put $x = 2009$

$P(2009) + 1 = (2009)!$

11. (a) : Let α be a root of $f(x)$, so we have $f(\alpha) = 0$ and thus $f(f(\alpha)) = 0$,

$\Rightarrow f(0) = 0 \Rightarrow \mu = 0$

We then have $f(x) = x(x+\lambda)$ and thus $\alpha = 0, -\lambda$.

$f(f(x)) = x(x+\lambda)(x^2+\lambda x+\lambda)$

We want λ such that $x^2 + \lambda x + \lambda$ has no real roots besides 0 and $-\lambda$. We can easily find that $0 \le \lambda < 4$.

12. (a) : We have $P(A) = P(7) = \frac{6}{36},\ P(B) = P(4) = \frac{3}{36}$

Since equal throws are disregarded,

Hence in each throw A is twice as likely to win as B.

Let $P(B) = p$, $P(A) = 2p$

$3p = 1 \Rightarrow P(B) = 1/3$

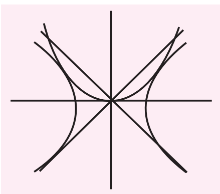
13. (a, c) : Observe that the lines L_1, L_2 and L_3 are parallel to the vector $(1, -1, -1)$.

Also $\Delta = 0 = \Delta_1$ and $b_1c_2 - b_2c_1 \ne 0$

$\because$ Coefficients are not proportional, the three planes intersect in a line

14. (c, d) : (α, α^2) lies on the parabola $y = x^2$

(α, α^2) must lie between the asymptotes of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ in 1st and 2nd quadrant



$\therefore$ Asymptotes are $y = \pm 2x$

$\therefore$ $2\alpha < \alpha^2 \Rightarrow \alpha < 0$ or $\alpha > 2$

and $-2\alpha < \alpha^2$

$\alpha < -2$ or $\alpha > 0$

$\therefore$ $\alpha \in (-\infty, -2)$ or $\alpha \in (2, \infty)$

15. (a, b, c) : The parabola can be written as

$$\left(x + \frac{2}{3}\right)^2 = 2\left(y - \frac{10}{9}\right)$$

i.e. $X^2 = 2Y\left(\text{where } X = x + \frac{2}{3}, Y = y - \frac{10}{9}\right)$.

A point on axis is $\left(\frac{-2}{3}, Y\right)$ from which three normals can be drawn if $Y > 0$.

16. (a, c) : Number of triplets $(3^r, 3^{r+1}, 3^{r+2})$ $(0 \le r \le n-2)$ is $n-1$

Number of triplets $(3^r, 3^{r+2}, 3^{r+4})(0 \le r \le n-4)$ is $n-3$

Number of triplets $\left(3^r, 3^{r+\frac{n-1}{2}}, 3^{r+n-1}\right)$ (n odd) is 2

and number of triplets $\left(3^r, 3^{r+\frac{n}{2}}, 3^{r+n}\right)$ (n even) is 1

If n is odd, the number of favourable outcomes

$$= (n-1) + (n-3) + \ldots + 4 + 2 = \frac{n^2-1}{4}$$

and if n is even, the number of favourable outcomes

$$= (n-1) + (n-3) + \ldots + 3 + 1 = \frac{n^2}{4}$$

$$\therefore \text{ Probability} = \frac{(n^2-1)/4}{{}^{(n+1)}C_3} = \frac{3}{2n} \text{ if } n \text{ is odd}$$

$$= \frac{n^2/4}{{}^{(n+1)}C_3} = \frac{3n}{2(n^2-1)} \text{ if } n \text{ is even}$$

17. (a, c) : $\frac{5^k+3}{4} = \frac{5^k-5+8}{4} = \frac{5(5^{k-1}-1)}{5-1} + 2$

$= 5 + 5^2 + 5^3 + \ldots (k-1)$ terms $+ 2$

So, if $k - 1$ = even, the last digit is 2

If $k - 1$ = odd, the last digit is $5 + 2 = 7$.

18. (b, c, d) : $\frac{x^3-6x^2+11x-6}{x^3+x^2-10x+8} = \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+4)}$

$\therefore$ $x \ne 1, 2, -4$ then $f(x) = \frac{x-3}{x+4}$

Range of $f(x) = R - \left\{1, -\frac{2}{5}, -\frac{1}{6}\right\}$

So equation does not have a solution if $\frac{a}{30} = -1, \frac{2}{5}, \frac{1}{6}$

$\Rightarrow$ $a = -30, 12, 5$

19. (a, b) : $\arg\frac{z_1}{z_2} = \pm\pi$ and

$|z_1 + i(z_3 - z_1)| = |z_1| + |z_3 - z_1|$ iff $\frac{z_1}{z_3 - z_1} = \frac{\pi}{2}$

Angle in a semicircle is 90°

So centre of circle $= \left(\frac{z_2+z_3}{2}\right)$ and ABC is right angle triangle.

20. (a, c) : $P(E \cap F) = P(E) \cdot P(F) = \frac{1}{6}$ ... (i)

$P(E^c \cap F^c) = (1 - P(E))(1 - P(F)) = \frac{1}{3}$

$\Rightarrow \quad P(E) + P(F) = \frac{5}{6}$ ... (ii)

$\Rightarrow \quad |P(E) - P(F)| = \frac{1}{6}$

As $(P(E) - P(F))(1 - P(F)) > 0$

$\Rightarrow \quad P(E) > P(F) \quad \Rightarrow \quad P(E) - P(F) = \frac{1}{6}$ ... (iii)

Solving (ii) and (iii) $\Rightarrow \quad P(E) = \frac{1}{2}, \; P(F) = \frac{1}{3}$

21. (b, c) : If the sides are a, a, b then the triangle is formed only when $2a > b$. So for any $a \in N$, b can change from 1 to $2a - 1$, where $a \le 1004$

$\Rightarrow$ no. of triangles $= 1 + 2 + 3 + ... + (2(1004) - 1)$
$= (1004)^2$

And if $1005 \le a \le 2008$, b can take any value from 1 to 2008

But a has 1004 possibilities hence

No. of triangles $= 1004 \times 2008 = 2(1004)^2$

$\therefore$ Total no. of isosceles triangles $= 3(1004)^2$

Remark : The above count of isosceles triangle also includes equilateral triangle.

(22-23):

Unit vector along line of intersection of S_1 and S_2

$$= \pm \frac{(2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} - \hat{k})}{|(2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} - \hat{k})|} = \pm \frac{(-\hat{i} + 3\hat{j} + 5\hat{k})}{\sqrt{35}}$$

22. (c) : $\pm 2\sqrt{35} \times \frac{(-\hat{i} + 3\hat{j} + 5\hat{k})}{\sqrt{35}} = \pm 2(-\hat{i} + 3\hat{j} + 5\hat{k})$

23. (d) : Equation of plane is

$-1(x - 1) + 3(y - 1) + 5(z - 1) = 0$

Distance $= \; S = \left| \frac{1 - 3 - 5}{\sqrt{35}} \right| = \frac{7}{\sqrt{35}}$

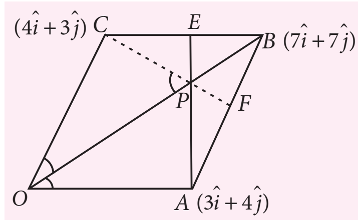
24. (d) : $\overrightarrow{OB} = 7\hat{i} + 7\hat{j}, \; \overrightarrow{OE} = 5\hat{i} + \frac{13}{3}\hat{j}, \; \overrightarrow{OP} = \frac{21}{5}(\hat{i} + \hat{j})$

25. (b) : Direction ratios of CP are proportional to $(1, 6, 0)$ then equation of line passing through $(2, 3, 4)$ and parallel to CP is

$$\frac{x - 2}{1} = \frac{y - 3}{6} = \frac{z - 4}{0}$$

26. (a) : The plane containing line AC and at a maximum distance from B must be perpendicular to the plane $OABC$. Since $OABC$ is rhombus, so $\overrightarrow{OB}$ must normal to the plane. So equation of required plane is

$[\vec{r} - 4\hat{i} - 3\hat{j}] \cdot (\hat{i} + \hat{j}) = 0 \quad \Rightarrow \quad \vec{r} \cdot (\hat{i} + \hat{j}) = 7$



27. (c) **28. (a)** **29. (d)**

Equation of hyperbola can be written as

$(x - 1)(y + 1) = 8$

Let $X = x - 1, \; Y = y + 1 \quad \Rightarrow \; XY = 8$

Equation of transverse axis is $Y = X$

$\Rightarrow \; x - 1 = y + 1 \Rightarrow x - y = 2$

$\therefore \quad P \equiv (4, 2)$

Let equation of parabola be $y = ax^2 + bx + c$

$2 = 16a + 4b + c$ ... (1)

Also $\left. \frac{dy}{dx} \right|_{(4,2)} = 0 \quad \Rightarrow \; 8a + b = 0$ ... (2)

Parabola passes through (1, 3)

$\Rightarrow \; 3 = a + b + c$ ... (3)

Solving (1), (2), (3) we get $a = \frac{1}{9}, b = -\frac{8}{9}, c = \frac{34}{9}$

So parabola is $9y = x^2 - 8x + 34$

Now tangent at P is $y = 2$

P' is $\left(\frac{11}{3}, \; 2 \right)$, C is $(1, -1)$

$$A = \frac{1}{2} \begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 11/3 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left[1(2 - 2) + 1\left(4 - \frac{11}{3}\right) + \left(8 - \frac{22}{3}\right) \right]$$

$= 1/2$ sq. units.

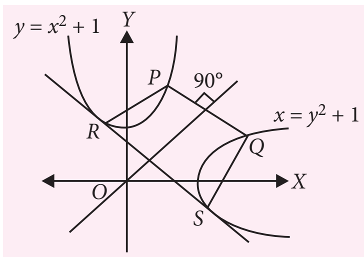
Let latus rectum is $y + x + \lambda = 0$

Now length of $\perp$ from centre $= \; 4\sqrt{2}$

So, $\frac{|\lambda|}{\sqrt{2}} \; = \; 4\sqrt{2} \; \Rightarrow \; \lambda = \pm 8$

So equation of latus rectum is $y + x - 8 = 0$.

30. (c) : PQ is common normal



$y = x^2 + 1, \; \frac{dy}{dx} = 2x = 1$

$x = \frac{1}{2},\ y = \frac{5}{4}$

As PQ is the shortest distance occur along common normal.

So, we can calculate P and Q

$P \equiv \left(\frac{1}{2}, \frac{5}{4}\right),\ Q \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$

Slope of common tangent is -1

Let equation of C.T is $y = -x + c$

Also $x^2 + x + 1 - c = 0$

$D = 0 \Rightarrow 1 - 4(1 - c) = 0,\ c = 3/4$

$\therefore$ Common tangent is $y = -x + \frac{3}{4} \Rightarrow 4x + 4y = 3$

Distance of common tangent from origin

$p = \frac{|-3|}{\sqrt{16+16}} = \frac{3}{4\sqrt{2}}$

31. (c) : $PQ = \sqrt{\frac{9}{16} + \frac{9}{16}} = \frac{3}{2\sqrt{2}}$

32. (b) : Also $R \equiv \left(-\frac{1}{2}, \frac{5}{4}\right),\ S \equiv \left(\frac{5}{4}, -\frac{1}{2}\right)$

$RS = \frac{7}{2\sqrt{2}}$, Distance between PQ and RS $(d) = \frac{1}{\sqrt{2}}$.

Area of $PQRS$ = $\frac{1}{\sqrt{2}} \times \frac{5}{\sqrt{2}} \times \frac{1}{2} = \frac{5}{4}$ sq. units

33. (b) : Let $R(h, k)$ be the point of intersection of the tangents to H at the extremities of the chord $L : x = 9$ then equation of L is $hx - ky = 9$

$\Rightarrow h = 1, k = 0.$

$\therefore$ Coordinates of R are $(1, 0)$.

Equation of the pair of tangents from R to H is

$(x^2 - y^2 - 9)(1 - 9) = (x - 9)^2 \qquad (SS_1 = T^2)$

$\Rightarrow 9x^2 - 8y^2 - 18x + 9 = 0.$

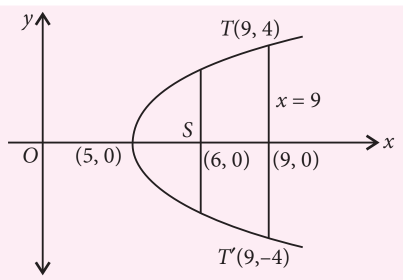
34. (b) : Now equation of the chord of contact of $R(1, 0)$ with respect to the parabola P: $y^2 = 4(x - 5)$ is

$y \times 0 = 2(x + 1) - 20 \Rightarrow x = 9$

35. (c) : Coordinates of T and T' are $(9, 4), (9, -4)$.

Coordinates of focus S of P are $(6, 0)$.

Area of $\Delta STT'$ = $4 \times 3 = 12$ sq. units.



36. (b) **37. (b)** **38. (b)**

Equation of any circle through the given points is

$(x - a)(x - 4a) + (y - 5a)(y - a) + \lambda a(4x + 3y - 19a) = 0$ for some $\lambda \in R$.

As it touches the y-axis, $\left(-3a + \frac{3a\lambda}{2}\right)^2 = 9a^2 - 19\lambda a^2$

Solving $\lambda = 0, \frac{-40}{9}$

The required circles are

$x^2 + y^2 - 5ax - 6ay + 9a^2 = 0,$

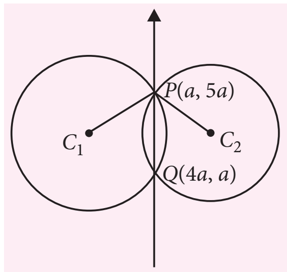
$x^2 + y^2 - 5ax - 6ay + 9a^2 - \frac{40a}{9}(4x + 3y - 19a) = 0$

Hence centres are $\left(\frac{5a}{2}, 3a\right)$ and $\left(\frac{205a}{18}, \frac{29a}{3}\right)$

The centres of the given circles are

$C_1\left(\frac{205a}{18}, \frac{29a}{3}\right)$

and $C_2\left(\frac{5a}{2}, 3a\right)$



Now the angle of intersection θ of these two circles is the angle between the radius vectors at the common point P to the two circles

i.e. $\angle C_1PC_2 = \theta$

Slope of $C_1P = \frac{\frac{29}{3}a - 5a}{\frac{205}{18}a - a} = \frac{84}{187}$

and slope of C_2P = $\frac{5a - 3a}{a - \frac{5a}{2}} = -\frac{4}{3}$

Now, $\tan\theta$ = $\frac{\frac{84}{187} + \frac{4}{3}}{1 - \frac{84}{187} \times \frac{4}{3}} = \frac{252 + 748}{561 - 336} = \frac{40}{9}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{40}{9}\right)$

Area of ΔOC_1C_2 = $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{205}{18} & \frac{29}{3} & 1 \\ \frac{5}{2} & 3 & 1 \end{vmatrix} a^2 = 5a^2$ sq. units

39. **(A) → (s); (B) → (r); (C) → (p); (D) → (r)**
Let $f(x) = \sin(\cos x) + \cos(\sin x)$

f is an even function. We can take $x \in \left[0, \frac{\pi}{2}\right]$.

In $\left[0, \frac{\pi}{2}\right]$, $\sin x$ is increasing and $\cos x$ is decreasing.

Hence f is a decreasing function. Therefore, maximum value of f is $f(0) = \sin 1 + 1$ and minimum value is $f(\pi/2) = 0 + \cos 1$.

Let $g(x) = \cos(\cos(\sin x))$. Obviously g is an even periodic function of period π. Hence g takes all of its values for $x \in \left[0, \frac{\pi}{2}\right]$.

It can be seen that g is an increasing function in $\left[0, \frac{\pi}{2}\right]$.

So maximum value of g is $g(\pi/2) = \cos(\cos 1)$, and minimum value of g is $g(0) = \cos 1$.

40. **(A) → (q); (B) → (r); (C) → (p); (D) → (s)**

(A) $\frac{dy}{dx} + (\tan x)y = \sec x$

$$\text{I.F.} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$
$$y \sec x = \int \sec^2 x dx = \tan x + c$$
$$y = \sin x + c \cos x$$

(B) $\frac{dy}{dx} + \frac{y}{x} = \sin x$, I.F. $= x$

$$yx = \int x \sin x dx + c = \sin x - x\cos x + c$$

(C) $\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{\cos x}{1+x^2}$

I.F. $= 1 + x^2$

$$y(1+x^2) = \int \cos x \, dx = \sin x + c$$

(D) $\frac{dy}{dx} + (2\tan x)y = \sin x$

I.F. $= \sec^2 x$

$$y \sec^2 x = \int \sin x \sec^2 x \, dx = \sec x + c$$
$$y = \cos x + c\cos^2 x$$

41. **(A) → (s); (B) → (r); (C) → (p); (D) → (q)**

(A) $x^3 - ax^2 + bx - 2 = 0$

Let x_1, x_2, x_3 be roots

A.M. ≥ G.M.

$$\Rightarrow \left(\frac{x_1+x_2+x_3}{3}\right)\left(\frac{\Sigma x_1x_2}{3}\right) \ge (x_1x_2x_3)^{1/3}(x_1x_2x_3)^{2/3}$$

$$\Rightarrow \frac{ab}{9} \ge 2 \Rightarrow (ab)_{\min} = 18$$

(B) $25200 = 2^4 3^2 5^2 7^1$

Now, if divisor is of the form $12\lambda + 6$ *i.e.* $6(2\lambda + 1)$ then there must be exactly one 2 and at least one 3.

∴ Number of divisors are $1 \cdot 2 \cdot (2 + 1)(1 + 1) = 12$

(C) Number of quadrilateral are $n(n - 5) = 8 \times 3 = 24$

(D) $[\cos x] = \tan x$

$\tan x = -1$ and $\cos x < 0$

∴ Three solutions.

42. **(A) → (q); (B) → (r); (C) → (s); (D) → (p)**

(A) Required number of selection

$$= {}^{2n}C_0 + {}^{2n}C_1 + \ldots + {}^{2n}C_n = 2^{2n-1} + \frac{1}{2}\frac{(2n)!}{(n!)^2}$$

(B) ${}^{n}C_4$ (each quadrilateral gives one point of intersection)

(C) $x_4 > x_3 > x_2 > x_1 > x_0$

${}^{10}C_5$ (5 distinct digits selection)

(D) Terms is involving 30, $3^{1/3}$, $3^{2/3} \to 3$

43. **(8) :** I_1 and I are excentre and incentre of ΔABC

$II_1 = AI_1 - AI$

$$= (r_1 - r)\operatorname{cosec}\frac{A}{2} = a\tan\frac{A}{2}\operatorname{cosec}\frac{A}{2} = \frac{a}{\cos(A/2)} = 8$$

44. **(6) :** Sum of area of ΔAPD and ΔBPC is equal to 1/2 of area of parallelogram $ABCD$.

45. **(4) :** The point E dividing $\overline{AB}$ externally in the ratio 2 : 1 is (9 , 0). Since P lies on the circle described on $\overline{DE}$ as diameter, coordinates of P are of the form $(5 + 4\cos\theta, 4\sin\theta)$

∴ maximum length of the altitude drawn from P to the base $AB = |4\sin\theta|_{\max} = 4$.

46. **(1) :** In expansion of $(a + b)^n$ the coefficient of 2nd, 3rd and 4th term are in A.P. which gives $n = 7$

Also $^7C_2\, a^5 b^2 = 84 \Rightarrow a^5b^2 = 4$

Now $\dfrac{\log_2 a^5 + \log_2 b^2}{2} \ge (\log_2 a^5 \cdot \log_2 b^2)^{1/2}$

$\Rightarrow \quad k \le \left(\dfrac{\log_2 a^5 b^2}{2}\right)^2$, where $k = \log_2 a^5 \cdot \log_2 b^2$

$k \le 1 \Rightarrow$ maximum value of k is 1.

47. (1) : $t_r = \sqrt{1+\dfrac{1}{r^2}+\dfrac{1}{(r+1)^2}} = \sqrt{\dfrac{r^2+(r+1)^2+r^2(r+1)^2}{r^2(r+1)^2}}$

$= \sqrt{\dfrac{2r^2+2r+1+r^2(r^2+2r+1)}{r^2(r+1)^2}}$

$= \sqrt{\dfrac{r^4+2r^3+3r^2+2r+1}{r^2(r+1)^2}} = \dfrac{r^2+r+1}{r(r+1)} = \dfrac{1}{r(r+1)}+1$

$= 1+\dfrac{1}{r}-\dfrac{1}{r+1}$, $S = 2000 - \dfrac{1}{2000}$, $|2000(S-2000)| = 1$

48. (1) : $\sum_{r=0}^{200} \alpha_r x^r = \sum_{r=0}^{200} \beta_r (1+x)^r$

$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{200}\, x^{200}$

$= \beta_0 + \beta_1(1+x) + \ldots. + \beta_{200}(1+x)^{200}$

Equating coefficient of x^{100}, we get

$\alpha_{100} = {}^{100}C_{100} + {}^{101}C_{100} + \ldots.. {}^{200}C_{100} = {}^{201}C_{101}$

Similarly we can find $\alpha_{100}, \ldots., \alpha_{200}$

$\sum_{r=100}^{200} \alpha_r = {}^{201}C_{101} + {}^{201}C_{102} + \ldots.. {}^{201}C_{201}$

$A = 2^{100}(2^{101} - 1)$

When A is divided by 15 remainder is 1.

49. (0) : Since $\left[\sqrt{2046}\right] = \left[\sqrt{2047}\right] = \left[\sqrt{2048}\right]$

$= \left[\sqrt{2049}\right] = 45$

$\therefore$ 2003$^{\text{rd}}$ term is 2003 + 45 = 2048

Hence remainder is 0

Remark : There is no perfect square between the numbers m^2 and $(m+1)^2$.

50. (7) : $f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots.. + f(\alpha^6 x)$

$= 7A_0 + \sum_{k=1}^{20} A_k x^k (1 + \alpha^k + \ldots\ldots + \alpha^{6k})$

but when $k \ne 7$ and $k \ne 14$, then

$1 + \alpha^k + \alpha^{2k} + \ldots. + \alpha^{6k} = 0$

Hence $f(x) + f(\alpha x) + \ldots + f(\alpha^6 x)$

$= 7A_0 + 7A_7\, x^7 + 7A_{14}\, x^{14} = 7(A_0 + A_7\, x^7 + A_{14}\, x^{14})$

$\Rightarrow \quad k = 7$

PRACTICE PAPER

BITSAT

Exam date : 14th to 28th May 2016

1. The area enclosed by $y^2 = 32x$ and $y = 4x$ is (in sq. units)
   (a) 8 (b) 4 (c) 4/3 (d) 8/3

2. If $f(x)=\begin{cases}\int_0^x (1+|1+t|)\,dt, & x>2\\ 5x-7, & x\le 2\end{cases}$, then
   (a) $f(x)$ is not continuous at $x = 2$
   (b) f is differentiable everywhere
   (c) RHL at $x = 2$ doesn't exist
   (d) f is continuous but not differentiable at $x = 2$

3. The value of $1+\sum_{k=0}^{14}\left\{\cos\frac{(2k+1)\pi}{15}+i\sin\frac{(2k+1)\pi}{15}\right\}$ is
   (a) 0 (b) –1 (c) 1 (d) i

4. If $(b - c)x^2 + (c - a)xy + (a - b)y^2$ is a perfect square, then a, b, c are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) none of these

5. The locus of the centre of a circle which touches the circle $|z - z_1| = a$ and $|z - z_2| = b$ externally (z, z_1 & z_2 are complex numbers) will be
   (a) an ellipse (b) a hyperbola
   (c) a circle (d) none of these

6. The solution of the differential equation $2x+\frac{dy}{dx}-y=3;$ given $y(0) = -1$ represents
   (a) straight line (b) circle
   (c) parabola (d) ellipse

7. The statement $P(n) : 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1$ is
   (a) true for all $n > 1$ (b) true for no n
   (c) true for all $n \in N$ (d) none of these

8. If $f(x) = x^n$, then the value of $f(1)+\frac{f^1(1)}{1!}+\frac{f^2(1)}{2!}+\ldots+\frac{f^n(1)}{n!}$, where $f^r(x)$ denotes the r^{th} order derivative of $f(x)$ with respect to x is
   (a) n (b) 2^n (c) 2^{n-1} (d) 1

9. The sum of the series $\frac{1}{3\times7}+\frac{1}{7\times11}+\frac{1}{11\times15}+\ldots..\infty$ is
   (a) 1/3 (b) 1/6 (c) 1/9 (d) 1/12

10. In an A.P. of which 1 is the first term, if the second, tenth and thirty fourth terms form a G.P., then the fourth term of the A.P. is
    (a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 2

11. The equation of the circle having its centre on the line $x + 2y - 3 = 0$ and passing through the point of intersection of the circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$ is
    (a) $x^2 + y^2 - 6x + 7 = 0$
    (b) $x^2 + y^2 - 3x + 4 = 0$
    (c) $x^2 + y^2 - 2x - 2y + 1 = 0$
    (d) $x^2 + y^2 + 2x - 4y + 4 = 0$

12. The equation of the circle passing through (1, 0) and (0, 1) and having smallest possible radius is
    (a) $x^2 + y^2 - x - y = 0$
    (b) $x^2 + y^2 + x + y = 0$
    (c) $x^2 + y^2 - 2x - y = 0$
    (d) $x^2 + y^2 - x - 2y = 0$

13. The volume of the greatest right circular cone, that can be described by the revolution about a side of a right angled triangle of hypotenuse 1 unit is
    (a) $\frac{2\pi}{3}$ cu. units (b) $\frac{2\pi}{3\sqrt{3}}$ cu. units
    (c) $\frac{2\pi}{9\sqrt{3}}$ cu. units (d) $\frac{4\pi}{9\sqrt{3}}$ cu. units

14. Let $R = \{(2, 3), (3, 4)\}$ be a relation defined on the set of natural numbers. The minimum number of ordered pairs required to be added in R so that enlarged relation becomes an equivalence relation is
    (a) 3 (b) 5 (c) 7 (d) 9

15. If the equations $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$ have both the roots common, then $2r - p = 0$
    (a) 2 (b) 1 (c) 0 (d) k

16. From a group of 20 persons, belonging to an association, a president, a secretary and three members are to be elected for the executive committee. The number of ways this can be done, is

(a) 310080 (b) 155040
(c) 620160 (d) 77520

17. The equation of the plane through the points (2, 2, 1) and (9, 3, 6) and $\perp$ to the plane $2x + 6y + 6z - 1 = 0$ is

(a) $3x + 4y + 5z = 9$ (b) $3x + 4y - 5z = 9$
(c) $3x + 4y - 5z + 9 = 0$ (d) none of these

18. The mean deviation from mean of the observations $a, a + d, a + 2d, ..., a + 2nd$ is

(a) $\frac{n(n+1)d^2}{3}$ (b) $\frac{n(n+1)}{2}d^2$
(c) $a+\frac{n(n+1)d^2}{2}$ (d) none of these

19. If the three consecutive coefficients in the expansion of $(1 + x)^n$ are 28, 56 and 70, then 'n' equals

(a) 6 (b) 4 (c) 8 (d) 10

20. The region represented by the inequalities $x \geq 6$, $y \geq 2$, $2x + y \leq 10$, $x \geq 0$, $y \geq 0$ is

(a) unbounded (b) a polygon
(c) exterior of a triangle (d) none of these

21. Given two numbers a and b. Let A denote the single A.M. and S denote the sum of n A.M.'s between a and b, then S/A depends on

(a) n, a, b (b) n, b (c) n, a (d) only n

22. $\int_0^{\pi/2} \frac{(\sin x+\cos x)^2}{\sqrt{1+\sin 2x}}dx =$

(a) 0 (b) 1 (c) 2 (d) 3

23. The gradient of the curve passing through (4, 0) is given by $\frac{dy}{dx}-\frac{y}{x}+\frac{5x}{(x+2)(x-3)}=0$ if the point $(5, a)$ lies on the curve, then the value of a is

(a) $\frac{67}{12}$ (b) $5\sin\frac{7}{12}$ (c) $5\log\frac{7}{12}$ (d) 0

24. The coefficient of x^4 in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is

(a) $\frac{405}{256}$ (b) $\frac{504}{259}$
(c) $\frac{450}{263}$ (d) none of these

25. A candidate appearing for the competitive test finds the following information on seeing the question paper:

(i) The total time is 3 hours.
(ii) The paper has two sections – section A and section B.
(iii) To answer atleast two questions from section A and atleast three questions from section B.
(iv) A question of first section carries 10 marks and that of section B carries 15 marks.
(v) Time to answer questions of section A is 15 minutes and that of section B is 25 minutes.
(vi) The maximum number of questions to be answered is 10.

The maximum marks, the candidate can secure is

(a) 115 (b) 125
(c) 100 (d) none of these

26. The length of the longest interval in which $f(x) = 3 \sin x - 4 \sin^3 x$ is increasing, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

27. The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then the value of b^2 is

(a) 1 (b) 5 (c) 7 (d) 9

28. Three persons A, B and C are to speak at a function along with 7 other persons. If the persons speak in a random order, the probability that A speaks before B and B speaks before C, is

(a) $\frac{7!-3!}{10!}$ (b) $\frac{{}^{10}C_3}{10!}$ (c) $\frac{3!}{10!}$ (d) $\frac{1}{6}$

29. Two teams are to play a series of 5 matches between them. A match ends in a win or loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches will contain 'n' people, where 'n' is

(a) 81 (b) 243
(c) 486 (d) none of these

30. An observer finds that the angular elevation of a tower is θ, on advancing 3 m towards, the elevation is 45° and on advancing 2 m nearer, the elevation is 90° – θ, the height of the tower is

(a) 1 m (b) 5 m (c) 6 m (d) 8 m

31. Let $F(x)=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$, where $\alpha \in R$

then $(F(\alpha)^{-1})$ is equal to

(a) $F(-\alpha)$ (b) $F(\alpha^{-1})$

(c) $F(2\alpha)$ (d) none of these

32. The lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ are perpendicular. Then

(a) $\frac{a}{a'}+\frac{c}{c'}=-1$ (b) $\frac{a}{a'}+\frac{c}{c'}=1$

(c) $aa' + cc' = -1$ (d) $aa' + cc' = 1$

33. Let $P(3, 2, 6)$ be a point in space and Q be a point on the line $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(-3\hat{i}+\hat{j}+5\hat{k})$. Then the value of μ for which the vector $\overline{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is

(a) 1/4 (b) – 1/4 (c) 1/8 (d) – 1/8

34. $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$ is equal to

(a) 1 (b) 5 (c) 10 (d) 15

35. The function $f : R \to R$ defined by $f(x) = (x - 1)(x - 2)(x - 3)$ is

(a) one-one but not onto

(b) onto but not one-one

(c) both one-one and onto

(d) neither one-one nor onto

36. Consider the sequence of numbers 121, 12321, 1234321, ... Each term in the sequence is

(a) a prime number

(b) square of an odd number

(c) divisible by 11

(d) form a G.P.

37. A factor of $\begin{vmatrix}a^2+x & ab & ac\\ ab & b^2+x & cb\\ ca & cb & c^2+x\end{vmatrix}$ is

(a) x^2 (b) $(a^2 + x)(b^2 + x)(c^2 + x)$

(c) $\frac{1}{x}$ (d) none of these

38. The least number of times a fair coin must be tossed so that the probability of getting at least one head is greater than 0.99, is

(a) 7 (b) 8 (c) 9 (d) 6

39. If $a\cos^2(3\alpha) + b\cos^4\alpha = 16\cos^6\alpha + 9\cos^2\alpha$ is an identity, then

(a) $a = 1, b = 24$ (b) $a = 3, b = 24$

(c) $a = 4, b = 2$ (d) $a = 7, b = 18$

40. The limit, when n tends to infinity, of the series $\frac{\sqrt{n}}{\sqrt{n^3}}+\frac{\sqrt{n}}{\sqrt{(n+4)^3}}+\frac{\sqrt{n}}{\sqrt{(n+8)^3}}+\frac{\sqrt{n}}{\sqrt{(n+12)^3}}+\ldots+\frac{\sqrt{n}}{\sqrt{[n+4(n-1)]^3}}$ is

(a) $\frac{5-\sqrt{5}}{10}$ (b) $\frac{2-\sqrt{2}}{2}$ (c) $5+\sqrt{5}$ (d) $2+\sqrt{2}$

41. The equation of one of the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ that is parallel to the line $x + 2y = 0$, is

(a) $x + 2y = 1$ (b) $x+2y=\frac{\pi}{2}$

(c) $x+2y=\frac{\pi}{4}$ (d) none of these

42. Let $f: (-1, 1) \to B$, $f(x)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ is one-one and onto. Then $B =$

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $\left(0,\frac{\pi}{2}\right)$

(c) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (d) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

43. The minimum value of $\left(1+\frac{1}{\sin^n\alpha}\right)\left(1+\frac{1}{\cos^n\alpha}\right)$ is

(a) 1 (b) 2

(c) $(1 + 2^{n/2})^2$ (d) none of these

44. If the line $y=x\sqrt{3}$ cuts the curve $x^3 + y^3 + 3xy + 5x^2 + 3y^2 + 4x + 5y - 1 = 0$ at the points A, B and C, then $OA\cdot OB\cdot OC$ (where O is origin) is

(a) $\frac{4}{13}(3\sqrt{3}-1)$ (b) $3\sqrt{3}+1$

(c) $\frac{1}{\sqrt{3}}(2+7\sqrt{3})$ (d) none of these

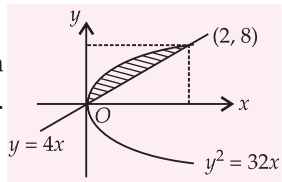
45. $\int 2^{2^{2^x}}\cdot 2^{2^x}\cdot 2^x\,dx$, is equal to

(a) $2^{2^{2^x}}\cdot(\ln 2)^3+C$ (b) $2^{2^x}\cdot(\ln 2)^2+C$

(c) $\frac{2^{2^{2^x}}}{(\ln 2)^3}+C$ (d) none of these

SOLUTIONS

1. (d): We have, $y^2 = 32x$ and $y = 4x$. So, intersection points are (0, 0) and (2, 8). So, area enclosed by $y = 4x$ and the parabola is



$$A=\int_0^2\left[\sqrt{32x}-4x\right]dx=\left[\sqrt{32}(x^{3/2})\times\frac{2}{3}-\frac{4x^2}{2}\right]_0^2$$

$$=\left[\sqrt{32}\times\sqrt{8}\times\frac{2}{3}-\frac{16}{2}\right]=\frac{8}{3}\text{sq. units}$$

2. **(d):** The function for $x > 2$ can be redefined as

$$f(x)=\int_0^1 1+(1-t)dt+\int_1^x 1+(t-1)dt$$

$$=2(1)-\frac{1}{2}+\frac{x^2}{2}-\frac{1}{2}=\frac{x^2}{2}+1$$

$$\text{R.H.L.}=\lim_{h\to 0}\left\{\frac{(2+h)^2}{2}+1\right\}=3$$

$$\text{L.H.L.}=\lim_{h\to 0}\{5(2-h)-7\}=3$$

$\because$ L.H.L = R.H.L $\Rightarrow$ f is continuous at $x = 2$

Now,

$$\text{L.H.D}=\lim_{h\to 0}\frac{f(2)-f(2-h)}{h}=\lim_{h\to 0}\frac{3-5(2-h)+7}{h}=5$$

$$\text{R.H.D}=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{\frac{(2+h)^2}{2}+1-3}{h}$$

$$=\lim_{h\to 0}\left\{\frac{h^2+4h+4+2-6}{2h}\right\}=\lim_{h\to 0}\left[\frac{h}{2}+2\right]=2$$

$\because$ L.H.D $\neq$ R.H.D, f is not differentiable at $x = 2$.

3. **(c):** $1+\sum_{k=0}^{14}\left\{\cos\frac{(2k+1)\pi}{15}+i\sin\frac{(2k+1)\pi}{15}\right\}$

$$=1+\sum_{k=0}^{14}e^{\frac{i(2k+1)\pi}{15}}=1+\sum_{k=0}^{14}\alpha^{2k+1},\quad\text{where }\alpha=e^{i\pi/15}$$

$$=1+(\alpha+\alpha^3+\alpha^5+\ldots+\alpha^{29})=1+\alpha\left(\frac{1-(\alpha^2)^{15}}{1-\alpha^2}\right)$$

$$=1+\alpha\left(\frac{1-\alpha^{30}}{1-\alpha^2}\right)=1+\alpha\left(\frac{1-1}{1-\alpha^2}\right)=1$$

$[\because \alpha^{30}=e^{i2\pi}=1]$

4. **(a):** For a perfect square, the discriminant = 0

$\therefore$ $(c-a)^2-4(b-c)(a-b)=0$

$\Rightarrow$ $(c+a)^2-4b(c+a)+4b^2=0 \Rightarrow (c+a-2b)^2=0$

$\Rightarrow$ $c+a=2b.$

Hence, a, b, c are in A.P.

5. **(b):** Let the circle $|z-z'|=c$, touches the circles $|z-z_1|=a$ and $|z-z_2|=b$ externally, then distance between centres = Sum of radii

$\therefore$ $|z'-z_1|=c+a$ ... (i)

and $|z'-z_2|=c+b$ ... (ii)

Subtracting (i) and (ii), we get

$|z'-z_1|-|z'-z_2|=a-b$

Hence locus of the centre of the assuming circle is

$|z-z_1|-|z-z_2|=a-b$

but $a-b<a+b$. Hence locus of the centre of the assuming circle is a hyperbola.

6. **(a):** The given differential equation can be written as

$$\frac{dy}{dx}-y=3-2x$$

This is a homogeneous linear differential equation.

$\therefore$ I.F. $=e^{-\int dx}=e^{-x}$

So, the solution is given by

$$ye^{-x}=\int(3-2x)e^{-x}dx$$

$\Rightarrow$ $y=-3+2(x+1)+c\,e^x$

$\therefore$ $y(0)=-3+2+c=-1$ $\Rightarrow$ $c=0$

$\therefore$ $y+3=2x+2$ is the solution which represents a straight line.

7. **(c):** $P(1): 1\times 1! = (1+1)! - 1$ is true.

Let $P(m)$ be true for some m

$\Rightarrow 1\times 1!+2\times 2!+\ldots+m\times m!=(m+1)!-1$

$\Rightarrow$ $1\times 1!+2\times 2!+\ldots+m\times m!+(m+1)\times(m+1)!$

$=(m+1)!-1+(m+1)\times(m+1)!$

$=(m+1)!\,(m+2)-1=(m+2)!-1$

$\Rightarrow$ $P(m+1)$ is also true

$\therefore$ $P(n)$ is true for all $n\in N$

8. **(b):** Given, $f(x)=x^n$

$$\text{So, } f^r(x)=\frac{n!}{(n-r)!}\cdot x^{n-r}\Rightarrow f^r(1)=\frac{n!}{(n-r)!}$$

$$\text{Now, } f(1)+\frac{f^1(1)}{1!}+\frac{f^2(1)}{2!}+\frac{f^3(1)}{3!}+\ldots+\frac{f^n(1)}{n!}$$

$$=\sum_{r=0}^{n}\frac{f^r(1)}{r!}=\sum_{r=0}^{n}\frac{n!}{(n-r)!r!}=\sum_{r=0}^{n}{}^nC_r=2^n.$$

9. **(d)**

10. **(d):** T_2, T_{10}, T_{34} are in G.P. *i.e.*, $1+d$, $1+9d$, $1+33d$ are in G.P.

$\Rightarrow$ $(1+9d)^2=(1+d)(1+33d)\Rightarrow(3d-1)d=0$

$\Rightarrow$ $d=\frac{1}{3}$

$\therefore$ $T_4=1+3d=1+1=2.$

11. **(a):** The equation of a circle passing through the intersection of two given circles is

$(x^2 + y^2 - 2x - 4y + 1) + \lambda(x^2 + y^2 - 4x - 2y + 4) = 0$

$$\Rightarrow x^2 + y^2 - 2x\left(\frac{1+2\lambda}{1+\lambda}\right) - 2y\left(\frac{2+\lambda}{1+\lambda}\right) + \left(\frac{1+4\lambda}{1+\lambda}\right) = 0 \quad ...(i)$$

Coordinates of the centre are $\left(\frac{1+2\lambda}{1+\lambda}, \frac{2+\lambda}{1+\lambda}\right)$.

Since the centre lies on $x + 2y - 3 = 0$.

$\therefore \; 1 + 2\lambda + 4 + 2\lambda - 3 - 3\lambda = 0 \Rightarrow \lambda = -2$

Putting $\lambda = -2$ in (i), we obtain the required circle $x^2 + y^2 - 6x + 7 = 0$.

12. (a) : Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

Its radius $(R) = \sqrt{g^2 + f^2 - c}$

Since (1) passes through (1, 0), (0,1)

$\therefore \; 1 + 2g + c = 0$ ...(2)

and $1 + 2f + c = 0$ ...(3)

(2) − (3) gives $g = f$ and $c = -(1 + 2g)$

$\therefore \; R = \sqrt{g^2 + g^2 + 1 + 2g} = \sqrt{2g^2 + 2g + 1}$

For R to be maximum or minimum

$$\frac{d(R^2)}{dg} = 0 \Rightarrow 4g + 2 = 0 \Rightarrow g = -\frac{1}{2}$$

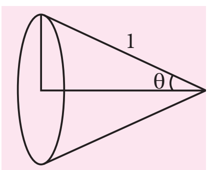
At $g = -\frac{1}{2}, \; \frac{d^2(R^2)}{dg^2} = 4 > 0$

$\therefore \; R$ is min. at $g = -\frac{1}{2}$

Also when $g = -\frac{1}{2}, \; f = -\frac{1}{2}$ and $c = 0$

$\therefore$ Required circle is $x^2 + y^2 - x - y = 0$

13. (c) : $V = \frac{1}{3}\pi \sin^2\theta \cos\theta$



For maximum/minimum $\frac{dV}{d\theta} = 0$

$$\Rightarrow 2\sin\theta\cos^2\theta = \sin^3\theta \Rightarrow \frac{\sin^2\theta}{2} = \frac{\cos^2\theta}{1}$$

$$\Rightarrow \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{2+1}{1}$$

$$\Rightarrow \frac{\cos^2\theta}{1} = \frac{\sin^2\theta + \cos^2\theta}{2+1} = \frac{1}{3}.$$

$$\therefore \; \text{Maximum } V = \frac{\pi}{3}\frac{2}{3}\frac{1}{\sqrt{3}} = \frac{2\pi}{9\sqrt{3}}$$

14. (c) : To make it reflexive. We need to add (2, 2), (3, 3), (4, 4). To make it symmetric (3, 2), (4, 3) to be added. To make it transitive (2, 4) and (4, 2) must be added. So, the relation $R = \{(2, 2), (3, 3), (4, 4), (2, 3), (3, 2), (3, 4), (4, 3), (2, 4), (4, 2)\}$

15. (c) : We have, $(6k + 2)x^2 + rx + 3k - 1 = 0$,
$(12k + 4)x^2 + px + 6k - 2 = 0$

Common roots $\Rightarrow \frac{6k+2}{12k+4} = \frac{r}{p} = \frac{3k-1}{6k-2}$

$$\Rightarrow \frac{r}{p} = \frac{1}{2} \Rightarrow 2r - p = 0$$

16. (a) : The president can be elected in ${}^{20}C_1$ ways, following which the secretary can be elected in ${}^{19}C_1$, and then the three members can be elected in ${}^{18}C_3$ ways.

$\therefore$ Required number of ways $= {}^{20}C_1 \cdot {}^{19}C_1 \cdot {}^{18}C_3$

$= 20 \cdot 19 \cdot \frac{18 \cdot 17 \cdot 16}{6} = 20 \cdot 19 \cdot 3 \cdot 17 \cdot 16 = 310080$

17. (b): The equation of a plane passing through (2, 2, 1) is

$a(x - 2) + b(y - 2) + c(z - 1) = 0$

This passes through (9, 3, 6) and is perpendicular to the plane $2x + 6y + 6z - 1 = 0$

$\therefore \; 7a + b + 5c = 0$ and $2a + 6b + 6c = 0$

$$\Rightarrow \frac{a}{-24} = \frac{b}{-32} = \frac{c}{40} \Rightarrow \frac{a}{3} = \frac{b}{4} = \frac{c}{-5}$$

So the required plane is

$3(x - 2) + 4(y - 2) - 5(z - 1) = 0$ or $3x + 4y - 5z = 9$

18. (d): $\bar{x} = \frac{1}{2n+1}[a + (a+d) + ... + (a+2nd)]$

$$= \frac{1}{2n+1}[(2n+1)a + d(1+2+...+2n)]$$

$$= a + d\frac{2n}{2}\cdot\frac{(2n+1)}{2n+1} = a + nd$$

$$\therefore \; \text{MD from mean} = \frac{1}{2n+1}\Sigma|x_i - \bar{x}|$$

$$= \frac{1}{2n+1}2|d|(1+2+...+n) = \frac{n(n+1)|d|}{(2n+1)}$$

19. (c) : We have, $\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{56}{28} \Rightarrow \frac{n-r}{r+1} = 2$

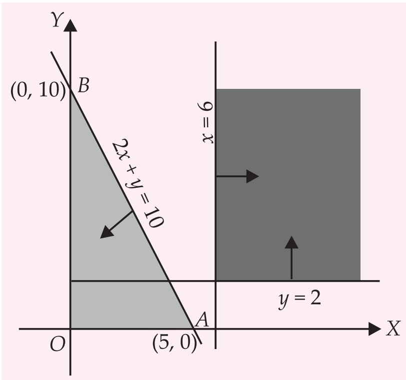
$\Rightarrow n = 3r + 2$

Also, $\frac{{}^nC_{r+2}}{{}^nC_{r+1}} = \frac{70}{56} \Rightarrow \frac{n-(r+1)}{r+2} = \frac{5}{4}$

$$\Rightarrow \frac{(3r+2)-(r+1)}{r+2} = \frac{5}{4}$$

$\therefore \; r = 2$ and $n = 8$

20. (d): The graph of the inequality $2x + y \le 10$ together with $x \ge 0$, $y \ge 0$ is the region of the triangle OAB (shown light shaded). The graph of the inequalities $x \ge 6$, $y \ge 2$ together with $x \ge 0$; $y \ge 0$ is the region (shown dark shaded). Since the two graphs have no point in common, the region represented by $x \ge 6$, $y \ge 2$, $2x + y \le 10$, $x \ge 0$, $y \ge 0$ is the empty set.



21. (d): We have, $A = \frac{a+b}{2}$. Let $A_1, A_2, ..., A_n$ be n A.M.'s between a and b. Then $a, A_1, A_2, ..., A_n, b$ is an A.P. with common difference $d = \frac{b-a}{n+1}$.

Now, $S = A_1 + A_2 + ... + A_n$

$$\Rightarrow \quad S = \frac{n}{2}(A_1 + A_n) = \frac{n}{2}(a+b) = nA \;\Rightarrow\; \frac{S}{A} = n$$

22. (c) : We have,

$$\int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)^2}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} (\sin x + \cos x) dx$$

$= 1 + 1 = 2.$

23. (c) : The differential equation is

$$\frac{dy}{dx} - \frac{y}{x} = -\frac{5x}{(x+2)(x-3)}$$

$$\text{I.F.} = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\log x} = \frac{1}{x}$$

$\therefore$ Solution is

$$y\left(\frac{1}{x}\right) = \int\left(\frac{1}{x}\right) \times \frac{-5x}{(x+2)(x-3)} dx = \log\left(\frac{x+2}{x-3}\right) + C$$

It passes through (4, 0), so $C = -\log 6$

$$\therefore \quad y = x\log\left\{\frac{(x+2)}{6(x-3)}\right\}$$

Putting $(5, a)$, we get $a = 5\log\left(\frac{7}{12}\right)$

24. (a) : $\binom{10}{r}\left(\frac{x}{2}\right)^{10-r}\left(-\frac{3}{x^2}\right)^r = \binom{10}{r} 2^{r-10}(-3)^r \cdot x^{10-3r}$

$\Rightarrow \; 10 - 3r = 4 \Rightarrow r = 2$

$\therefore$ The coefficient of x^4 is $\binom{10}{2} 2^{-8} \cdot 9 = \frac{405}{256}$.

25. (a) : Let x_1, x_2 denote the number of questions to be attempted from section A and B respectively. Then mathematically the problem can be written as

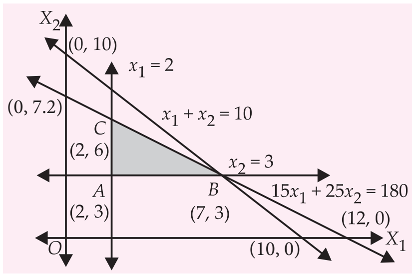
Maximize $M = 10x_1 + 15x_2$

subject to constraints : $x_1 \ge 2$, $x_2 \ge 3$

$15x_1 + 25x_2 \le 180$

$x_1 + x_2 \le 10$ and $x_1 \ge 0$, $x_2 \ge 0$

Convert the inequations into equations and draw the graph as shown in the figure.



The shaded part shows the feasible region. The corner points of the feasible region are $A(2, 3)$, $B(7, 3)$, $C(2, 6)$.

At $A(2, 3)$, $M = 10(2) + 15(3) = 65$

At $B(7, 3)$, $M = 10(7) + 15(3) = 115$

At $C(2, 6)$, $M = 10(2) + 15(6) = 110$

Since the maximum value of M occurs at B

$\therefore$ $x_1 = 7$, $x_2 = 3$ is the optimal solution and 115 is the maximum marks.

26. (a) : $f(x) = \sin 3x \;\Rightarrow\; f'(x) = 3\cos 3x > 0$

$\cos x > 0$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos 3x > 0$ in $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$.

The length of the interval is $\frac{\pi}{3}$.

27. (c) : For the given ellipse $a^2 = 16$. Therefore,

$$e = \sqrt{1 - \frac{b^2}{a^2}} \;\Rightarrow\; e = \sqrt{1 - \frac{b^2}{16}} = \frac{\sqrt{16-b^2}}{4}.$$

So, the foci of the ellipse are $(\pm ae, 0)$

i.e., $\left(\pm\sqrt{16-b^2},\, 0\right)$

For the hyperbola $a^2 = \left(\frac{12}{5}\right)^2$, $b^2 = \left(\frac{9}{5}\right)^2$.

The eccentricity e is given by

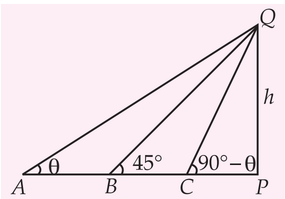
$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{81}{144}}=\frac{15}{12}=\frac{5}{4}.$$

Thus, the foci of the hyperbola are $(\pm\, ae, 0)$ or $(\pm\, 3, 0)$. Since the foci of the ellipse and hyperbola coincide, therefore $\sqrt{16-b^2}=3 \Rightarrow b^2=7$

28. (d): For any arrangement of the 7 persons (other than A, B, C), the three persons A, B and C can be arranged in 3! ways out of which there is only one way in which A comes before B and B comes before C.
Hence, required probability is (1/3!).

29. (b): The smallest number of people
= Total number of possible forecasts
= Total number of possible results
$= 3 \times 3 \times 3 \times 3 \times 3 = 243$

30. (c) : Let PQ be the tower of height h.
Here, $AB = 3$ m and $BC = 2$ m (see figure)



$\Rightarrow 3 = AB = PA - PB = h\cot\theta - h$
and $2 = BC = PB - CP = h - h\tan\theta$
so that $\frac{h-2}{h}=\frac{h}{h+3} \Rightarrow h = 6\,\text{m}.$

31. (a) : We have,

$$F(\alpha)F(-\alpha)=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I \quad \therefore \quad F(-\alpha) = [F(\alpha)]^{-1}$$

32. (c) : The lines are $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$ and

$$\frac{x-b'}{a'}=\frac{y}{1}=\frac{z-d'}{c'}$$

The d.r.'s are a, 1, c and a', 1, c'.
The lines are perpendicular if $aa' + 1 + cc' = 0$
$\Rightarrow aa' + cc' = -1.$

33. (a) : A point on the line is $Q(-3\mu + 1, \mu - 1, 5\mu + 2)$
The direction ratios of PQ are
$< -3\mu - 2, \mu - 3, 5\mu - 4>$

As $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z - 1 = 0$ we have, $(-3\mu - 2) - 4(\mu - 3) + 3(5\mu - 4) = 0$
$\Rightarrow 8\mu = 2$
$\therefore \mu = 1/4.$

34. (d): The given expression is equal to
$1 + [\tan(\tan^{-1} 2)]^2 + 1 + [\cot(\cot^{-1} 3)]^2$
$= 1 + 4 + 1 + 9 = 15$

35. (b): We have $f(x) = (x - 1)(x - 2)(x - 3)$.
Since $1, 2, 3 \in R$ and $f(1) = 0, f(2) = 0, f(3) = 0$, f cannot be one – one.
Let $k \in R$. $f(x) = k$ implies $(x - 1)(x - 2)(x - 3) = k$.
$\Rightarrow x^3 - 6x^2 + 11x - 6 - k = 0.$
Since a cubic equation has at least one real root, there exist some $x' \in R$ such that $f(x') = k$.
$\therefore f$ is onto.

36. (b): The n^{th} term of the sequence is
$123 \ldots n(n + 1)n \ldots 321$
$= 1 \times 10^{2n} + 2 \times 10^{2n-1} + 3 \times 10^{2n-2} + \ldots + n \times 10^{n+1} + (n + 1) \times 10^n + n \times 10^{n-1} + \ldots + 3 \times 10^2 + 2 \times 10 + 1$

Let, $S_1 = 1 \times 10^{2n} + 2 \times 10^{2n-1} + \ldots + n \times 10^{n+1}$ ...(1)

$$\frac{1}{10}S_1 = 1\times 10^{2n-1}+\ldots+(n-1)\times 10^{n-1}+n\times 10^n \quad \text{...(2)}$$

Subtracting (2) from (1), we get

$$\frac{9}{10}S_1 = 1\times 10^{2n}+1\times 10^{2n-1}+\ldots+1\times 10^{n+1}-n\times 10^n$$

$$S_1=\frac{10}{9}[10^{2n}+10^{2n-1}+\ldots+10^{n+1}-n\times 10^n]$$

$$S_1=\frac{10}{9}\left[\frac{10^{2n}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}}-n\times 10^n\right]$$

$$=\left(\frac{10}{9}\right)^2\left[10^{2n}\left(1-\frac{1}{10^n}\right)\right]-\left[\left(\frac{10}{9}\right)n\times 10^n\right]$$

Again let
$S_2 = (n + 1) \times 10^n + n \times 10^{n-1} + \ldots + 2 \times 10 + 1$ ...(3)

$$\frac{1}{10}S_2=(n+1)\times 10^{n-1}+\ldots+3\times 10+2+\frac{1}{10} \quad \text{...(4)}$$

Subtracting (4) from (3), we get

$$\frac{9}{10}S_2=(n+1)\times 10^n-\left[10^{n-1}+10^{n-2}+\ldots+1+\frac{1}{10}\right]$$

$$=(n+1)\times 10^n-\frac{10^{n-1}\left(1-\frac{1}{10^{n+1}}\right)}{1-\frac{1}{10}}$$

$$\Rightarrow S_2 = \frac{10}{9}(n+1)\times 10^n - \left(\frac{10}{9}\right)^2 10^{n-1}\left(1-\frac{1}{10^{n+1}}\right)$$

$\therefore$ The n^{th} term $= S_1 + S_2$

$$= \left(\frac{10}{9}\right)^2\left[10^{2n}\left(1-\frac{1}{10^n}\right) - 10^{n-1}\left(1-\frac{1}{10^{n+1}}\right)\right]$$

$$+\frac{10}{9}[-n\times 10^n + (n+1)\times 10^n]$$

$$= \left(\frac{10}{9}\right)^2\left[10^{2n} - 2\cdot 10^{n-1} + \frac{1}{10^2}\right]$$

$$= \left\{\frac{10}{9}\left[10^n - \frac{1}{10}\right]\right\}^2 = \left[\frac{10^{n+1}-1}{9}\right]^2$$

$$= \left[\frac{999...(n+1)\text{ times}}{9}\right]^2$$

$= [111 \ldots (n + 1)$ times$]^2 =$ (odd number)^2.

37. (a) : We have,

$$\Delta = \begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{abc}\begin{vmatrix} a^3+ax & a^2b & a^2c \\ ab^2 & b^3+bx & b^2c \\ ac^2 & bc^2 & c^3+cx \end{vmatrix}$$

Applying $C_1 \to C_1/a$, $C_2 \to C_2/b$, $C_3 \to C_3/c$

$$\Delta = \begin{vmatrix} a^2+x & a^2 & a^2 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = (a^2+b^2+c^2+x)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

$= (a^2 + b^2 + c^2 + x)\,\{(b^2x + c^2x + x^2) - (b^2x) + (-c^2x)\}$

$= x^2\,(a^2 + b^2 + c^2 + x)$. So, x^2 is a factor of Δ

38. (a) : The probability of getting at least one head ≥ 0.99

$\therefore$ The probability of getting no head ≤ 0.01

$$\Rightarrow \frac{1}{2^n} \leq \frac{1}{100} \Rightarrow 2^n \geq 100 \Rightarrow n = 7.$$

39. (a) : $16(\cos\alpha)^6 + 9(\cos\alpha)^2 = (4\cos^3\alpha)^2 + 9\cos^2\alpha$

$= (\cos 3\alpha + 3\cos\alpha)^2 + 9\cos^2\alpha$

$= (\cos 3\alpha)^2 + 18\cos^2\alpha + 6\cos\alpha \cdot \cos 3\alpha$

$= (\cos 3\alpha)^2 + 18\cos^2\alpha + 3(\cos 2\alpha + \cos 4\alpha)$

$= (\cos 3\alpha)^2 + 18\cos^2\alpha + 3\cos 4\alpha + 3(2\cos^2\alpha - 1)$

$= (\cos 3\alpha)^2 + 24\cos^2\alpha + 3(\cos 4\alpha - 1)$

$= (\cos 3\alpha)^2 + 24\cos^2\alpha - 6\sin^2 2\alpha$

$= (\cos 3\alpha)^2 + 24(\cos\alpha)^4$

$= a(\cos 3\alpha)^2 + b(\cos\alpha)^4$ (given)

On comparing, we get $a = 1$, $b = 24$

40. (a) : The general term is

$$T_{r+1} = \frac{\sqrt{n}}{\sqrt{[n+4r]^3}},\ r = 0, 1, 2, ..., n-1$$

$$= \frac{\sqrt{n}}{n^{3/2}\sqrt{\left[1+\frac{4r}{n}\right]^3}} = \frac{1}{n}\cdot\frac{1}{\left(1+\frac{4r}{n}\right)^{3/2}}$$

Hence the series is $\displaystyle\sum_{r=0}^{n-1} T_{r+1} = \sum_{r=0}^{n-1}\frac{1}{n}\frac{1}{\left(1+\frac{4r}{n}\right)^{3/2}}$

The required limit is $\displaystyle S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\left(1+\frac{4r}{n}\right)^{3/2}}$

$$= \int_0^1 \frac{1}{(1+4x)^{3/2}}dx = \int_0^1 (1+4x)^{-3/2}dx$$

$$= \left[\frac{(1+4x)^{-1/2}}{-\frac{1}{2}\times 4}\right]_0^1 = \frac{1}{10}(5-\sqrt{5})$$

41. (b): We have $y = \cos(x + y)$ ....(i)

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\sin(x+y)\left(1+\frac{dy}{dx}\right) \quad ...(ii)$$

Since the tangent is parallel to $x + 2y = 0$, therefore

the slope of the tangent $= -\frac{1}{2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$.

Putting $\frac{dy}{dx} = -\frac{1}{2}$ in (ii) we get

$$-\frac{1}{2} = -\sin(x+y)\left(1-\frac{1}{2}\right) \Rightarrow \sin(x+y) = 1 \quad ...(iii)$$

Squaring and adding (i) and (iii), we get

$y^2 + 1 = 1 \Rightarrow y^2 = 0 \Rightarrow y = 0$

Putting $y = 0$ in (i) and (iii), we get

$\sin x = 1$ and $\cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

Thus, the points on the curve (i), where tangents are parallel to $x + 2y = 0$ are

$$\left(\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(-\frac{3\pi}{2}, 0\right)$$

The equation of the tangent at $\left(\frac{\pi}{2}, 0\right)$ is

$$y - 0 = -\frac{1}{2}\left(x - \frac{\pi}{2}\right) \Rightarrow x + 2y = \frac{\pi}{2}$$

42. (c) : $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x$

$$B = f(-1, 1) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

43. (c) : Let $f(\alpha) = \left(1 + \frac{1}{\sin^n \alpha}\right)\left(1 + \frac{1}{\cos^n \alpha}\right)$

$$= 1 + \frac{1}{\sin^n \alpha} + \frac{1}{\cos^n \alpha} + \frac{1}{\sin^n \alpha \cos^n \alpha}$$

Then, $f'(\alpha) = -\frac{n}{\sin^{n+1} \alpha}\cos\alpha + \frac{n\sin\alpha}{\cos^{n+1}\alpha}$

$$-\frac{n}{\sin^{n+1}\alpha\cos^{n+1}\alpha}\{\cos^2\alpha - \sin^2\alpha\}$$

For maximum/minimum $f'(\alpha) = 0$

$\Rightarrow \cos\alpha = \sin\alpha \Rightarrow \alpha = \pi/4$

Now, $f(\alpha)$ is maximum at $\alpha = 0$ and $\alpha = \pi/2$ and between two maxima there is one minima. Hence $\alpha = \pi/4$ gives the minimum value of $f(\alpha)$ and is given by $f\left(\frac{\pi}{4}\right) = \left(1 + 2^{n/2}\right)^2$.

44. (a) : Putting $y = x\sqrt{3}$ in the equation of curve

$$x^3(3\sqrt{3}+1) + x^2(3\sqrt{3}+14) + (4+5\sqrt{3})x - 1 = 0$$

If the roots of this equation are x_1, x_2 and x_3

$$\Rightarrow x_1x_2x_3 = \frac{1}{3\sqrt{3}+1} = \frac{3\sqrt{3}-1}{26}$$

Also, for a point A, coordinates are $(x_1, x_1\sqrt{3})$

$$\therefore OA = \sqrt{x_1^2 + 3x_1^2} = 2x_1$$

$$\Rightarrow OA \cdot OB \cdot OC = (2x_1)(2x_2)(2x_3) = 8x_1x_2x_3$$

$$= \frac{4}{13}(3\sqrt{3}-1)$$

45. (c) : Putting $2^{2^{2x}} = z \Rightarrow 2^{2^{2x}} \cdot 2^{2x} \cdot 2^x (\ln 2)^3 dx = dz$,

$$\text{Now, } I = \int (2^{2^{2x}} \cdot 2^{2^x} \cdot 2^x)dx = \int \frac{dz}{(\ln 2)^3} = \left[\frac{z}{(\ln 2)^3} + C\right]$$

$$= \frac{2^{2^{2^x}}}{(\ln 2)^3} + C.$$

ACE YOUR WAY CBSE XI

Series 1

SETS

HIGHLIGHTS

SET

A well defined collection of objects.

Types of Respresentation of sets

Roster/Tabular form: The elements are separated by commas and are enclosed within braces { }.

Set builder form : The elements are represented by using a variable x followed by ':' and after ':' we write the characteristic property possessed by the elements. The whole description is enclosed within braces { }.

TYPES OF SETS

Empty Set - Set which does not contain any element. It is denoted by ϕ or { }.

Finite Set - Set which is empty or having finite number of elements.

Infinite Set - Set which is not a finite set.

Equal Sets - Two given sets having exactly the same elements.

SUBSET

For any two sets P and Q,

- P is said to be the subset of Q, *i.e.* $P \subseteq Q$ if every element of P is also an element of Q.
- If $P \subset Q$ and $P \neq Q$, then P is called proper subset of Q and Q is called superset of P.

Subsets of set of real numbers

$N \subset Z \subset Q$, $Q \subset R$, $T \subset R$, $N \not\subset T$

where, N = set of natural number.

Z = set of integers,

Q = set of rational numbers,

R = set of real numbers and

T = set of irrational numbers.

Intervals of subsets of R

Let $a, b, \in R$ and $a < b$, then

- **Closed Interval :** $[a, b] = \{x : a \leq x \leq b\}$
- **Semi closed or semi open interval :**
  $[a, b) = \{x : a \leq x < b\}$ and $(a, b] = \{x : a < x \leq b\}$
- **Open interval :** $(a, b) = \{x : a < x < b\}$

POWER SET

Collection of all subsets of any set. It is denoted by $P(A)$ for any set A.

UNIVERSAL SET

Any set which is superset of all the sets. Denoted by U.

VENN DIAGRAM

A diagram used to illustrate relationship between sets.

OPERATIONS ON SETS

(i) Union of sets, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

(ii) Intersection of sets, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

(iii) Difference of sets, $A - B = \{x : x \in A, x \notin B\}$

(iv) Complement of a set, $A' = U - A$, where U is the universal set.

LAWS OF ALGEBRA OF SETS

For three sets A, B and C

(i) Idempotent law
  (a) $A \cup A = A$ (b) $A \cap A = A$

(ii) Identity law
  (a) $A \cup \phi = A$ (b) $A \cap U = A$

(iii) Commutative law
  (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$

(iv) Associative law
  (a) $(A \cup B) \cup C = A \cup (B \cup C)$
  (b) $A \cap (B \cap C) = (A \cap B) \cap C$

(v) Distributive law
  (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(vi) De-Morgan's law
  (a) $(A \cup B)' = A' \cap B'$
  (b) $(A \cap B)' = A' \cup B'$

(vii) Law of empty set and universal set
  (a) $A \cup U = U$ (b) $A \cap \phi = \phi$
  (c) $U' = \phi$ (d) $\phi' = U$
  Complement laws
  (e) $A \cap A' = \phi$ (f) $A \cup A' = U$
  Double complement law
  (g) $(A')' = A$

IMPORTANT FORMULAE BASED ON NUMBER OF ELEMENTS IN SETS

For two finite sets A and B, we have

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, if $A \cap B \neq \phi$
- $n(A \cup B) = n(A) + n(B)$, if $A \cap B = \phi$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

Very Short Answer Type

1. Represent the set $A = \{-3, -2, -1, 0, 1, 2, 3, 4\}$ in the set builder form.

2. Are the following sets equal?
$A = \{x : x$ is a letter in the word 'wolf'$\}$
$B = \{x : x$ is a letter in the word 'follow'$\}$
$C = \{x : x$ is a letter in the word 'flow'$\}$

3. If X and Y are two sets such that $n(X) = 17$, $n(Y) = 23$ and $n(X \cup Y) = 38$, find $n(X \cap Y)$.

4. Write the following subset of R as an interval :
$\{x : x \in R, -4 < x \leq 6\}$

5. Write the following set in the set builder form :
$Q = \{1, 4, 9, 16\}$

6. Find the power sets of the following set :
$C = \{1, 2, \{3, 4\}\}$

7. Find the union of the following pair of sets :
$A = \{x : x$ is a natural number and $1 < x \leq 6\}$ and
$B = \{x : x$ is a natural number and $6 < x < 10\}$

8. Prove that : If $A \subseteq B$ and $x \notin B$, then $x \notin A$.

Long Answer Type - I

9. Let A and B be two sets. Prove that :
$(A - B) \cup B = A$ if and only if $B \subset A$

10. If A and B are any two sets, then prove that
$A \cap B = A \cup B \Rightarrow A = B$

11. In a survey of 400 students in a school, 100 were listed as drinking apple juice, 150 as drinking orange juice and 75 were listed as drinking apple as well as orange juice. Find how many students were drinking neither apple juice nor orange juice?

12. (i) If $A \cap B' = \phi$, then show that $A \subset B$.
(ii) If $A' \cup B = U$, then show that $A \subset B$.

13. Let A, B and C be three sets such that $A \cup B = C$ and $A \cap B = \phi$, then prove that $A = C - B$.

14. For any set A, B, C prove that
$A - B = B - A$ iff $A = B$

15. If universal set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6\}$, $C = \{2, 3, 7\}$, then find :
(i) $(A - B)'$ (ii) $B' - A'$ (iii) $(A \cap C)'$

Long Answer Type - II

16. A class has 175 students. Following is the description showing the number of students studying one or more of the following subjects in this class.
Mathematics 100, Physics 70, Chemistry 46, Mathematics and Physics 30, Mathematics and Chemistry 28, Physics and Chemistry 23, Mathematics, Physics and Chemistry 18. How many students are enrolled in Mathematics alone, Physics alone and Chemistry alone?

17. Prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$

18. A college awarded 38 medals in Football, 15 in Basketball and 20 in Cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports?

19. If A and B are any two sets, then prove that
$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

20. In a town of 10,000 families it was found that 40% families buy newspaper A, 20% families buy newspaper B, 10% families buy newspaper C, 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% buy all the three newspapers, find
(i) the number of families who buy newspaper A only.
(ii) the number of families who buy none of the newspaper A, B and C.

SOLUTIONS

1. Let x denote an arbitrary element of A.
Then x can be any integer from -3 to 4.
$\therefore$ $A = \{x : x \text{ is an integer and } -3 \le x \le 4\}$.

2. $A = \{x : x = w, x = o, x = l, x = f\} = \{w, o, l, f\}$
$B = \{x : x = f, x = o, x = l, x = w\} = \{f, o, l, w\}$
$C = \{x : x = f, x = l, x = o, x = w\} = \{f, l, o, w\}$
$\Rightarrow$ $A = B = C$.

3. $\because$ $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$
$\Rightarrow$ $38 = 17 + 23 - n(X \cap Y)$
$\Rightarrow$ $n(X \cap Y) = 40 - 38 = 2$

4. We have,
$\{x : x \in R, -4 < x \le 6\} = (-4, 6]$

5. Here, $Q = \{1, 4, 9, 16\}$.
Note that each of the number in the set is a perfect square.
$\therefore$ $Q = \{x : x \text{ is square of a natural number and } x \le 16\}$
$\Rightarrow$ $Q = \{x : x \text{ is square of a natural number less than } 5\}$

6. $P(C) = \{X : X \subseteq C\}$
$= \{\phi, \{1\}, \{2\}, \{\{3,4\}\}, \{1,2\}, \{1, \{3,4\}\}, \{2, \{3,4\}\}, \{1, 2, \{3,4\}\}\}$

7. Here $A = \{x : x \text{ is a natural number and } 1 < x \le 6\}$
$= \{2, 3, 4, 5, 6\}$
and $B = \{x : x \text{ is a natural number and } 6 < x < 10\}$
$= \{7, 8, 9\}$
$\therefore$ $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9\}$

8. Let $A \subseteq B$ and $x \notin B$
To prove : $x \notin A$
If possible, let $x \in A \Rightarrow x \in B$ $[\because A \subseteq B]$
This contradicts the given condition that $x \notin B$.
Hence $x \notin A$.

9. First, let $(A - B) \cup B = A$. Then, we have to prove that $B \subset A$.
Now, $(A - B) \cup B = A$
$\Rightarrow$ $(A \cap B') \cup B = A$ $[\because A - B = A \cap B']$
$\Rightarrow$ $(A \cup B) \cap (B' \cup B) = A$
$\Rightarrow$ $(A \cup B) \cap U = A$
$\Rightarrow$ $A \cup B = A$
$\Rightarrow$ $B \subset A$
Conversely, let $B \subset A$. Then, we have to prove that $(A - B) \cup B = A$.
Now, $(A - B) \cup B = (A \cap B') \cup B$
$= (A \cup B) \cap (B' \cup B) = (A \cup B) \cap U$
$= A \cap U = A$ $[\because B \subset A \therefore A \cup B = A]$

10. $x \in A \Rightarrow x \in A$ or $x \in B \Rightarrow x \in A \cup B \Rightarrow A \subset A \cup B$.
$x \in A \cap B \Rightarrow x \in A$ and $x \in B \Rightarrow x \in A \Rightarrow A \cap B \subset A$
Similarly, we can show that $B \subset A \cup B$ and $A \cap B \subset B$
Now $A \subset A \cup B$ and $A \cup B = A \cap B$,
$\therefore$ $A \subset A \cap B$ ... (i)
Also $A \cap B \subset A$ ... (ii)
From (i) and (ii), we get $A \cap B = A$ ... (iii)
Similarly, we get $A \cap B = B$ ... (iv)
From (iii) and (iv), we have $A = B$.

11. Let U = set of students surveyed.

A = set of people drinking apple juice.

B = set of people drinking orange juice.

Given, $n(U) = 400$, $n(A) = 100$, $n(B) = 150$ and $n(A \cap B) = 75$

Now $n(A' \cap B') = n(A \cup B)'$

$= n(U) - n(A \cup B)$

$= n(U) - [n(A) + n(B) - n(A \cap B)]$

$= 400 - 100 - 150 + 75 = 225.$

$\therefore$ Number of students drinking neither apple juice nor orange juice is 225.

12. (i) $A = A \cap U$, where U is the universal set

$= A \cap (B \cup B') = (A \cap B) \cup (A \cap B')$

$= (A \cap B) \cup \phi \quad [\because A \cap B' = \phi] = A \cap B$

Hence, $A = A \cap B \Rightarrow A \subset B$.

(ii) $B = B \cup \phi = B \cup (A \cap A')$

$= (B \cup A) \cap (B \cup A') = (B \cup A) \cap U = B \cup A$

Hence, $B = B \cup A \Rightarrow A \subset B$

13. Here $A \cup B = C$

$\therefore \ (A \cup B) - B = C - B$

$\Rightarrow (A \cup B) \cap B' = C - B \quad (\because A - B = A \cap B')$

$\Rightarrow (A \cap B') \cup (B \cap B') = C - B$

$\Rightarrow (A \cap B') \cup \phi = C - B$

$\Rightarrow (A \cap B') = C - B$

$\Rightarrow A = C - B \quad (\because A \cap B = \phi)$

14. Only If part : Let $A - B = B - A$ ... (1)

Let $x \in A \Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in B)$

$\Leftrightarrow x \in (A - B)$ or $x \in (A \cap B)$

$\Leftrightarrow x \in (B - A)$ or $x \in (A \cap B)$ [From (1)]

$\Leftrightarrow (x \in B$ and $x \notin A)$ or $(x \in B$ and $x \in A)$

$\Leftrightarrow x \in B$

Thus, $A - B = B - A \Rightarrow A = B$.

Hence $A = B$

If part : Let $A = B$

Now $A - B = A - A = \phi$

and $B - A = A - A = \phi \quad [\because B = A]$

$\therefore \ A - B = B - A$

Thus, $A = B \Rightarrow A - B = B - A$

15. (i) $A - B = \{x : x \in A$ and $x \notin B\} = \{1, 4\}$.

$\therefore \ (A - B)' = S - (A - B) = S - \{1, 4\}$

$= \{0, 2, 3, 5, 6, 7, 8, 9\}$

(ii) $B' = S - B = S - \{2, 3, 5, 6\} = \{0, 1, 4, 7, 8, 9\}$

$A' = S - A = S - \{1, 2, 3, 4\} = \{0, 5, 6, 7, 8, 9\}$

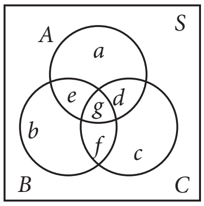
$B' - A' = \{0, 1, 4, 7, 8, 9\} - \{0, 5, 6, 7, 8, 9\} = \{1, 4\}$

(iii) $A \cap C = \{1, 2, 3, 4\} \cap \{2, 3, 7\} = \{2, 3\}$

$\therefore \ (A \cap C)' = S - (A \cap C) = \{0, 1, 4, 5, 6, 7, 8, 9\}$

16. Let A, B and C denote the sets of students studying Mathematics, Physics and Chemistry respectively.

Let us denote the number of elements (students) contained in the bounded region as shown in the diagram by a, b, c, d, e, f and g respectively.



Then, $a + d + g + e = 100$

$b + f + g + e = 70$

$c + f + g + d = 46$

$g + e = 30$

$g + d = 28$

$g + f = 23$

$g = 18$

Solving these, we get $g = 18$, $f = 5$, $d = 10$, $e = 12$, $c = 13$, $b = 35$, $a = 60$.

$\therefore \ a + b + c + d + e + f + g = 153.$

Number of students studying Mathematics only $= a = 60$

Number of students studying Physics only $= b = 35$

Number of students studying Chemistry only $= c = 13$

17. Let $x \in A \cap (B - C)$

$\Rightarrow x \in A$ and $x \in (B - C)$

$\Rightarrow x \in A$ and $(x \in B$ and $x \notin C)$

$\Rightarrow (x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow x \in (A \cap B)$ and $x \notin (A \cap C)$

$\Rightarrow x \in (A \cap B) - (A \cap C)$

Hence $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$ ...(i)

Now, let $y \in (A \cap B) - (A \cap C)$

$\Rightarrow y \in (A \cap B)$ and $y \notin (A \cap C)$

$\Rightarrow (y \in A$ and $y \in B)$ and $(y \in A$ and $y \notin C)$

$\Rightarrow y \in A$ and $(y \in B$ and $y \notin C)$

$\Rightarrow y \in A$ and $y \in (B - C)$

$\Rightarrow y \in A \cap (B - C)$

$\therefore \ (A \cap B) - (A \cap C) \subseteq A \cap (B - C)$ ...(ii)

$\therefore$ From (i) and (ii), we have

$A \cap (B - C) = (A \cap B) - (A \cap C)$

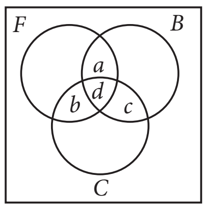
18. Consider the Venn diagram

Let a = number of men who got medals in Football and Basketball only

b = number of men who got medals in Football and Cricket only

c = number of men who got medals in Basketball and Cricket only

d = number of men who got medals in all the three sports Football, Basketball and Cricket



Given, $n(F \cup B \cup C) = 58$, $n(F) = 38$, $n(B) = 15$, $n(C) = 20$, $n(F \cap B \cap C) = d = 3$

Now, $n(F \cup B \cup C) = n(F) + n(B) + n(C) - n(F \cap B) - n(B \cap C) - n(F \cap C) + n(F \cap B \cap C)$

Rvp

$\therefore$ $58 = 38 + 15 + 20 - n(F \cap B) - n(B \cap C) - n(F \cap C) + 3$

$\Rightarrow$ $n(F \cap B) + n(B \cap C) + n(F \cap C) = 18$

$\Rightarrow$ $a + d + c + d + b + d = 18$

$\Rightarrow$ $a + b + c = 18 - 3d = 18 - 3 \times 3 = 9$

$\therefore$ Number of people who got medals in exactly two of the three sports = 9

19. Let x be an arbitrary element of $(A - B) \cup (B - A)$.

Then, $x \in (A - B) \cup (B - A)$

$\Rightarrow$ $x \in A - B$ or $x \in B - A$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \notin B$ or $x \notin A)$

$\Rightarrow$ $x \in (A \cup B)$ and $x \notin (A \cap B)$

$\Rightarrow$ $x \in (A \cup B) - (A \cap B)$

$\therefore$ $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$ ... (i)

Again, let y be an arbitrary element of $(A \cup B) - (A \cap B)$.

Then, $y \in (A \cup B) - (A \cap B)$

$\Rightarrow$ $y \in A \cup B$ and $y \notin A \cap B$

$\Rightarrow$ $(y \in A$ or $y \in B)$ and $(y \notin A$ and $y \notin B)$

$\Rightarrow$ $(y \in A$ and $y \notin B)$ or $(y \in B$ and $y \notin A)$

$\Rightarrow$ $y \in (A - B)$ or $y \in (B - A)$

$\Rightarrow$ $y \in (A - B) \cup (B - A)$

$\therefore$ $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$ ...(ii)

Hence, from (i) and (ii), we have

$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

20. If P, Q, R denote the sets of families who buy newspaper A, B and C respectively, then

$$n(P) = \frac{40}{100} \times 10000 = 4000,$$

$$n(Q) = \frac{20}{100} \times 10000 = 2000,$$

$$n(R) = \frac{10}{100} \times 10000 = 1000,$$

$$n(P \cap Q) = \frac{5}{100} \times 10000 = 500,$$

$$n(Q \cap R) = \frac{3}{100} \times 10000 = 300,$$

$$n(P \cap R) = \frac{4}{100} \times 10000 = 400,$$

and $n(P \cap Q \cap R) = \frac{2}{100} \times 10000 = 200$

$n(P \cup Q \cup R) = n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(R \cap P) + n(P \cap Q \cap R)$

Hence, $n(P \cup Q \cup R) = 4000 + 2000 + 1000 - 500 - 300 - 400 + 200 = 6000$

(i) Number of families who buy newspaper A only

$= n(P \cap Q' \cap R')$

$= n(P) - n(P \cap Q) - n(P \cap R) + n(P \cap Q \cap R)$

$= 4000 - 500 - 400 + 200 = 3300$

(ii) Number of families who buy none of newspaper A, B and $C = n((P \cup Q \cup R)')$

$= 10000 - n(P \cup Q \cup R)$

$= 10000 - 6000 = 4000$

ACE YOUR WAY CBSE XII

Series 1

RELATIONS AND FUNCTIONS

HIGHLIGHTS

RELATIONS

For any two non-empty sets A and B, every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

TYPES OF RELATIONS

A relation R on a set A is said to be

(i) **Empty relation,** if no element of A is related to any element of A.

(ii) **Universal relation,** if each element of A is related to every element of A.

(iii) **Identity relation,** if every element of A is related to itself only.

(iv) **Reflexive relation,** if $(a, a) \in R\ \forall\ a \in A$

(v) **Symmetric relation,** if $(a, b) \in R \Rightarrow (b, a) \in R$ $\forall a, b \in A$

(vi) **Transitive relation,** if $(a, b) \in R$ and $(b, c) \in R$ $\Rightarrow (a, c) \in R\ \forall\ a, b, c \in A$

(vii) **Equivalence relation,** if it is reflexive, symmetric and transitive.

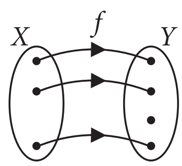
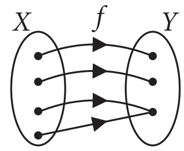
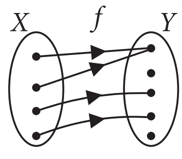
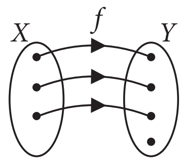
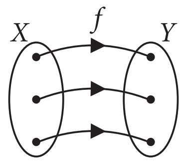
Equivalence Class

Equivalence class $[a]$ containing $a \in X$ for an equivalence relation R in X is the subset of X containing all elements b related to a.

FUNCTIONS

A relation f from X to Y is a function if each element of X has an image in Y and no element of X has more than one image in Y.

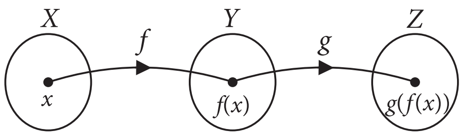
TYPES OF FUNCTIONS

Type	Definition	Representation
One-one (Injective) Function	A function $f : X \to Y$ is one-one, if different elements of X have different images in Y under f.	
Onto (Surjective) Function	A function $f : X \to Y$ is onto, if every element of Y is the image of some element of X under f.	
Many-one Function	A function $f : X \to Y$ is many-one, if two or more than two elements of X have the same image in Y.	
Into Function	A function $f : X \to Y$ is into, if there exists a single element in Y having no pre-image in X.	
Bijective Function	A function is bijective, if it is both one-one and onto.	

COMPOSITION OF FUNCTIONS

Composition of two functions $f : X \to Y$ and $g : Y \to Z$, denoted by gof and is defined as

$$gof\,(x) = g(f(x))\ \forall\ x \in X$$



INVERSE OF A FUNCTION

Inverse of a function $f: X \to Y$ is defined as

$$f^{-1}: Y \to X \text{ s.t. } f^{-1}(y) = x \Leftrightarrow f(x) = y$$

A function f is invertible iff f is one-one onto.

BINARY OPERATIONS

Let A be a non-empty set. A binary operation '$*$' on set A is a function from $A \times A$ to A.

- A binary operation $*$ on a set A is

(i) **Commutative,** if $a * b = b * a, \ \forall \ a, b \in A$

(ii) **Associative,** if $(a * b) * c = a * (b * c), \forall \ a, b, c \in A$

(iii) **Distributive,** over another binary operation 'o' on A,

if $a * (boc) = (a * b) \text{ o } (a * c)$

and $(boc) * a = (b * a) \text{ o } (c * a), \ \forall \ a, b, c \in A$

Identity element and Inverse of an element

Let $*$ be a binary operation on a set A. Then,

- an element $e \in A$ is the **identity element** for $*$, if $a * e = a = e * a, \ \forall \ a \in A$
- let $e \in A$ be the identity element. An element $a \in A$ is said to be **inverse of an element** $b \in A$, if $b * a = e = a * b$.

Very Short Answer Type

1. Let $f: R \to R$ and $g: R \to R$ be defined by $f(x) = x^2$, $g(x) = x + 2; \ \forall \ x \in R$ (set of all real numbers). Is $gof = fog$?

2. Let $*$ be a binary operation on R, the set of all real numbers, defined by $a * b = \sqrt{a^2 + b^2}$ for all $a, b \in R$. Show that $*$ is commutative.

3. If $f(x)$ is an invertible function, then find the inverse of $f(x) = \frac{3x-2}{5}$.

4. Let $A = \{1, 2, 3\}$ and $R = \{(a, b) : a, b \in A, a$ divides b and b divides $a\}$. Show that R is an identity relation on A.

5. Let $f: Z \to Z$ be defined by $f(x) = x + 2$. Find $g: Z \to Z$ such that $gof = I_Z$.

Long Answer Type-I

6. Let $A = Q \times Q$ and let $*$ be a binary operation on A defined by
$(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$
Then, with respect to $*$ on A
(i) Find the identity element in A.
(ii) Find the invertible elements of A.

7. Find the value of parameter α for which the function $f(x) = 1 + \alpha x, \alpha \neq 0$ is the inverse of itself.

8. Let $A = R - \left\{\frac{3}{5}\right\}$ and $B = R - \left\{\frac{7}{5}\right\}$.

Let $f: A \to B$; $f(x) = \frac{7x+4}{5x-3}$ and $g: B \to A$; $g(y) = \frac{3y+4}{5y-7}$. Show that $(gof) = I_A$ and $(fog) = I_B$.

9. If Q is the set of rational numbers and R is a relation defined on Q by $xRy \Leftrightarrow |x - y| \leq \frac{1}{2}$, then prove that R is not an equivalence relation.

10. Let $f: N \to Y$, where $f(x) = 4x^2 + 12x + 15$ and $Y = \text{range}(f)$. Show that f is invertible and find f^{-1}.

Long Answer Type-II

11. Prove that the intersection of two equivalence relations is also an equivalence relation.

12. Let $f(x) = [x]$ and $g(x) = |x|$. Find

(i) $(gof)\left(\frac{-5}{3}\right) - (fog)\left(\frac{-5}{3}\right)$

(ii) $(gof)\left(\frac{5}{3}\right) - (fog)\left(\frac{5}{3}\right)$

(iii) $(f + 2g)(-1)$

13. Let '$*$' be a binary operation on N given by $a * b = \text{L.C.M.}(a, b)$ for all $a, b \in N$.
(i) Find $5 * 7$, $20 * 16$
(ii) Is $*$ commutative?
(iii) Is $*$ associative?
(iv) Find the identity element in N.
(v) Which elements of N are invertible? Find them.

14. If $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$, show that $fof(x) = x$ for all $x \neq \frac{2}{3}$. What is the inverse of f?

15. Let $A = \{x \in Z : 0 \le x \le 12\}$

Show that $R = \{(a, b) : |a - b|$ is a multiple of $4\}$ is

(i) reflexive (ii) symmetric

(iii) transitive

Find the set of elements related to 1.

SOLUTIONS

1. $(gof)(x) = g[f(x)] = g(x^2) = x^2 + 2$

$(fog)(x) = f[g(x)] = f(x+2) = (x+2)^2$

Hence, $gof \ne fog$.

2. We have,

$a * b = \sqrt{a^2 + b^2} = \sqrt{b^2 + a^2} = b * a$ for all $a, b \in R$

$\Rightarrow$ $a * b = b * a$ for all $a, b \in R$

So, $*$ is commutative on R.

3. Let $y = f(x) = \frac{3x-2}{5}$, then $D_f = R$ and $R_f = R$

$\Rightarrow$ $5y = 3x - 2$ $\Rightarrow$ $5y + 2 = 3x$

$\Rightarrow$ $x = \frac{5y+2}{3}$, $\forall\, x, y \in R$

$\Rightarrow$ $f^{-1}(y) = \frac{5y+2}{3}, \forall x, y \in R$

$\therefore$ $f^{-1}(x) = \frac{5x+2}{3}$

4. Given, $A = \{1, 2, 3\}$

$a, b \in A$, a divides b and b divides $a \Rightarrow a = b$

$\therefore$ $R = \{(a, a), a \in A\} = \{(1, 1), (2, 2), (3, 3)\}$

Hence, R is the identity relation on A.

5. Given $f(x) = x + 2$

Now, $gof = I_Z \Rightarrow (gof)(x) = I_Z(x)$, for all $x \in Z$

$\Rightarrow$ $g(f(x)) = x$, for all $x \in Z$

$\Rightarrow$ $g(x + 2) = x$, for all $x \in Z$

$\Rightarrow$ $g(x) = x - 2$, for all $x \in Z$ [Replace x by $x - 2$]

Hence the required function $g : Z \to Z$ is given by $g(x) = x - 2$, for all $x \in Z$.

6. (i) Let (x, y) be the identity element in A such that $(a, b) \in A$. Then,

$(a, b) * (x, y) = (a, b) = (x, y) * (a, b)$

$\Rightarrow$ $(ax, b + ay) = (a, b) = (xa, y + xb)$

$\Rightarrow$ $(ax, b + ay) = (a, b)$ and $(a, b) = (xa, y + xb)$

$\Rightarrow$ $ax = a$ and $b + ay = b$ and $xa = a, y + xb = b$

$\Rightarrow$ $x = 1, y = 0$

Clearly, $(1, 0) \in Q \times Q = A$

So, $(1, 0)$ is the identity element in A.

(ii) Let (a, b) be an invertible element of A. Then there exists $(c, d) \in A$ such that

$(a, b) * (c, d) = (1, 0) = (c, d) * (a, b)$

$\Rightarrow$ $(ac, b + ad) = (1, 0)$ and $(ca, d + cb) = (1, 0)$

$\Rightarrow$ $ac = 1, b + ad = 0$ and $ca = 1, d + cb = 0$

$\Rightarrow$ $c = \frac{1}{a}$ and $d = -\frac{b}{a}$, if $a \ne 0$

Thus, (a, b) is an invertible element of A, if $a \ne 0$ and in such a case the inverse of (a, b) is $\left(\frac{1}{a}, \frac{-b}{a}\right)$.

7. Clearly, $f(x)$ is a bijective function from R to itself.

$\therefore$ $fof^{-1}(x) = x$ $\Rightarrow f(f^{-1}(x)) = x$

$\Rightarrow$ $1 + \alpha f^{-1}(x) = x$ $\Rightarrow f^{-1}(x) = \frac{x-1}{\alpha}$

It is given that $f(x) = f^{-1}(x)$ for all $x \in R$

$\Rightarrow$ $1 + \alpha x = \frac{x-1}{\alpha}$ for all $x \in R$

$\Rightarrow$ $\alpha x + 1 = \left(\frac{1}{\alpha}\right)x + \left(\frac{-1}{\alpha}\right)$ for all $x \in R$

$\Rightarrow$ $\alpha = \frac{1}{\alpha}$ and $1 = -\frac{1}{\alpha}$

$\Rightarrow$ $\alpha^2 = 1$ and $\alpha = -1 \Rightarrow \alpha = \pm 1$ and $\alpha = -1$

Thus, $\alpha = -1$

Hence, the required value of α is -1.

8. Let $x \in A$.

Then, $(gof)(x) = g[f(x)]$

$$= g\left(\frac{7x+4}{5x-3}\right) = \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7}$$

$$= \frac{21x+12+20x-12}{35x+20-35x+21} = \frac{41x}{41} = x = I_A(x)$$

$\therefore$ $(gof) = I_A$

Again, let $y \in B$. Then, $(fog)(y) = f[g(y)]$

$$= f\left(\frac{3y+4}{5y-7}\right) = \frac{7\left(\frac{3y+4}{5y-7}\right)+4}{5\left(\frac{3y+4}{5y-7}\right)-3}$$

$$= \frac{21y+28+20y-28}{15y+20-15y+21} = \frac{41y}{41} = y = I_B(y)$$

$\therefore$ $(fog) = I_B$

Hence, $(gof) = I_A$ and $(fog) = I_B$.

9. (i) $|x-x|=0\le\frac{1}{2};\forall x\in Q.$

$\therefore$ xRx *i.e.*, R is reflexive on Q.

(ii) $xRy \Rightarrow |x-y|\le\frac{1}{2}$

$\Rightarrow |y-x|\le\frac{1}{2}$ $\qquad [\because |y-x|=|x-y|]$

$\Rightarrow$ yRx *i.e.* R is symmetric on Q.

(iii) Let $x=\frac{1}{4}, y=\frac{3}{5}, z=1.$ Then

$$|x-y|=\left|\frac{1}{4}-\frac{3}{5}\right|=\frac{7}{20}\le\frac{1}{2}$$

$$|y-z|=\left|\frac{3}{5}-1\right|=\frac{2}{5}\le\frac{1}{2}$$

$$|x-z|=\left|\frac{1}{4}-1\right|=\frac{3}{4}\not\le\frac{1}{2}$$

$\therefore$ $\frac{1}{4}R\frac{3}{5}$ and $\frac{3}{5}R1$ but $\frac{1}{4}$ is not R-related to 1.

So R is not transitive. Thus R is not an equivalence relation on Q.

10. Let $x_1, x_2 \in N$ such that

$f(x_1) = f(x_2) \Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$\Rightarrow$ $4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0$

$\Rightarrow$ $(x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$

$\Rightarrow$ $(x_1 - x_2)(x_1 + x_2 + 3) = 0$

$\Rightarrow$ $x_1 - x_2 = 0$ $\qquad [x_1, x_2 \in N \therefore x_1 + x_2 + 3 \ne 0]$

$\Rightarrow$ $x_1 = x_2$ $\qquad \therefore f$ is one-one

Also, range $(f) = Y$. So, f is onto

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, for f being onto, there exists $x \in N$ such that $y = f(x)$

Now, $y = f(x) \Rightarrow y = 4x^2 + 12x + 15$

$\Rightarrow$ $y = (2x + 3)^2 + 6 \Rightarrow (2x + 3) = \sqrt{y-6}$

$\Rightarrow$ $x = \frac{1}{2}\left(\sqrt{y-6}-3\right)$

$\Rightarrow$ $f^{-1}(y) = \frac{1}{2}\left(\sqrt{y-6}-3\right)$

Thus, we define $f^{-1} : Y \to N$ as

$$f^{-1}(y) = \frac{1}{2}\left(\sqrt{y-6}-3\right)$$

11. Let R and S be two equivalence relations on a set A. Then both are reflexive, symmetric and transitive.

(i) $(x, x) \in R$ and $(x, x) \in S, \forall x \in A$

$\Rightarrow$ $(x, x) \in R \cap S \ \forall x \in A$

$\Rightarrow$ $R \cap S$ is reflexive.

(ii) Let $(x, y) \in R \cap S \Rightarrow (x, y) \in R$ and $(x, y) \in S$

$\Rightarrow$ $(y, x) \in R$ and $(y, x) \in S$ $[\because R, S$ are symmetric$]$

$\Rightarrow$ $(y, x) \in R \cap S$

Thus $R \cap S$ is symmetric.

(iii) Let $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$

$\Rightarrow$ $[(x, y) \in R$ and $(x, y) \in S]$ and $[(y, z) \in R$ and $(y, z) \in S]$

$\Rightarrow$ $[(x, y) \in R$ and $(y, z) \in R]$ and $[(x, y) \in S$ and $(y, z) \in S]$

$\Rightarrow$ $(x, z) \in R$ and $(x, z) \in S$ $[\because R$ and S are transitive$]$

$\Rightarrow$ $(x, z) \in R \cap S$.

Hence $R \cap S$ is transitive.

Thus $R \cap S$ is reflexive, symmetric and transitive and therefore it is an equivalence relation on A.

12. We have, $f(x) = [x]$ and $g(x) = |x|$

Clearly, Domain $(f) = R$ and, Domain $(g) = R$.

Therefore, each of fog, gof and $f + 2g$ has domain R.

(i) $$(gof)\left(\frac{-5}{3}\right)-(fog)\left(\frac{-5}{3}\right)=g\left\{f\left(\frac{-5}{3}\right)\right\}-f\left\{g\left(\frac{-5}{3}\right)\right\}$$

$$=g\left\{\left[\frac{-5}{3}\right]\right\}-f\left\{\left|\frac{-5}{3}\right|\right\}$$

$$=g(-2)-f\left(\frac{5}{3}\right)=|-2|-\left[\frac{5}{3}\right]=2-1=1$$

(ii) $$(gof)\left(\frac{5}{3}\right)-(fog)\left(\frac{5}{3}\right)=g\left\{f\left(\frac{5}{3}\right)\right\}-f\left\{g\left(\frac{5}{3}\right)\right\}$$

$$=g\left\{\left[\frac{5}{3}\right]\right\}-f\left\{\left|\frac{5}{3}\right|\right\}=g(1)-f\left(\frac{5}{3}\right)$$

$$=|1|-\left[\frac{5}{3}\right]=1-1=0$$

(iii) $(f + 2g)\ (-1) = f\ (-1) + 2g\ (-1) = [-1] + 2\ |-1|$
$= -1 + 2 \times 1 = 1.$

13. (i) We have,

$a * b =$ LCM of a and b

$\therefore$ $5 * 7 =$ (LCM of 5 and 7) $= 35$

and $20 * 16 =$ (LCM of 20 and 16) $= 80$

(ii) We have,

$a * b =$ LCM of a and b

and $b * a =$ LCM of b and a

We know that for any $a, b \in N$

LCM of a and b = LCM of b and a

$\therefore$ $a * b = b * a$

So, $*$ is commutative on N.

(iii) For any $a, b, c \in N$, we have
$(a * b) * c$ = LCM of $(a * b)$ and c
= LCM of (LCM of a and b) and c
= LCM of a, b and c

Similarly, we have
$a * (b * c)$ = LCM of a, b and c

$\therefore$ $(a * b) * c = a * (b * c)$, for all $a, b, c \in N$

So, $*$ is associative on N.

(iv) Let $e \in N$ be the identity element. Then,
$a * e = a = e * a$, for all $a \in N$

$\Rightarrow$ $a * e = a$, for all $a \in N$

$\Rightarrow$ LCM $(a, e) = a$, for all $a \in N$

$\Rightarrow$ $e = 1$

So, 1 is the identity element in N

(v) Let a be an invertible element in N. Then, there exists $b \in N$ such that $a * b = 1 = b * a$

$\Rightarrow$ LCM $(a, b) = 1$ = LCM (b, a)

$\Rightarrow$ $a = b = 1$

Thus, 1 is the only invertible element of N.

14. Given $f(x) = \dfrac{4x+3}{6x-4},\ x \neq \dfrac{2}{3}$ ...(i)

Since $x \neq \dfrac{2}{3}$, $\therefore$ Domain of $f = R - \left\{\dfrac{2}{3}\right\}$

Now $fof(x) = f[f(x)] = f\left(\dfrac{4x+3}{6x-4}\right)$

$$= \frac{4\left(\dfrac{4x+3}{6x-4}\right)+3}{6\left(\dfrac{4x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x$$

Let $y = f(x) \Rightarrow y = \dfrac{4x+3}{6x-4} \Rightarrow x = \dfrac{4y+3}{6y-4}$

$\therefore$ For x to be real, $6y - 4 \neq 0$ $\therefore y \neq \dfrac{2}{3}$

Hence, range of $f = R - \left\{\dfrac{2}{3}\right\}$

Thus, $f : R - \left\{\dfrac{2}{3}\right\} \to R - \left\{\dfrac{2}{3}\right\}$

We have shown that $fof(x) = x$

Hence $f^{-1} = f$ $\therefore$ $f^{-1}(x) = f(x) = \dfrac{4x+3}{6x-4}$

15. Clearly, $A = \{0, 1, 2, 3, 4, ..., 10, 11, 12\}$

Here, R satisfies the following properties

(i) Let a be an arbitrary element of A. Then,
$a - a = 0$, which is a multiple of 4

$\therefore$ $a\,R\,a$ for all $a \in A$

So, R is reflexive.

(ii) Let $a, b \in A$, such that
$aRb \Rightarrow |a - b|$ is a multiple of 4
$\Rightarrow |-(a - b)|$ is a multiple of 4
$\Rightarrow |b - a|$ is a multiple of 4
$\Rightarrow b\,R\,a$

$\therefore$ R is symmetric.

(iii) Let $a, b, c \in A$ such that
$a\,R\,b \Rightarrow |a - b|$ is a multiple of 4
and $b\,R\,c \Rightarrow |b - c|$ is a multiple of 4
Let $|a - b| = 4k_1$ and $|b - c| = 4k_2$. Then,
$\Rightarrow a - b = \pm 4k_1$, and $b - c = \pm 4k_2$
$\therefore a - c = (a - b) + (b - c) = \pm 4k_1 \pm 4k_2$
$\Rightarrow a - c$ is a multiple of 4.
$\Rightarrow |a - c|$ is a multiple of 4.
$\therefore$ $a\,R\,b$, $b\,R\,c \Rightarrow a\,R\,c$. So, R is transitive.

Thus, R is reflexive, symmetric and transitive.
Hence, R is an equivalence relation.

Now, $[1] = \{x \in A : x\,R1\}$
$= \{x \in A : |x - 1|$ is a multiple of $4\}$
$= \{1, 5, 9\}$

Hence, the required set is $\{1, 5, 9\}$.

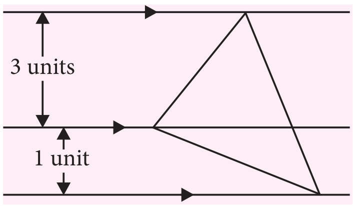
OLYMPIAD CORNER

1. Let a, b and c be positive real numbers such that $abc = 1$. Prove that :
$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\le 1$$

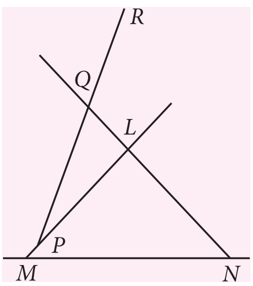
2. Find all positive integers k for which the following statement is true : if $F(x)$ is a polynomial with integer coefficients satisfying the condition
$0 \le F(c) \le k$ for $c = 0, 1, ..., k + 1$,
then $F(0) = F(1) = ... = F(k + 1)$.

3. Find all pairs of integers (a, b) such that the polynomial $ax^{17} + bx^{16} + 1$ is divisible by $x^2 - x - 1$.

4. The vertices of an equilateral triangle lie on three parallel lines which are 3 units and 1 unit apart as shown. Find the area (in square units) of this triangle.



5. In the diagram, angles LMN and LNM are 45 degrees. The intervals LM, LN, PQ and QR are each 1 unit long. What is the furthest distance that R can be from the line MN?



SOLUTIONS

1. **1st solution :** Setting $x = a + b$, $y = b + c$ and $z = c + a$, the inequality becomes
$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\le 1$$
$$i.e.,\ \frac{1}{y+1}+\frac{1}{z+1}\le\frac{x}{x+1}$$
$$i.e.,\ \frac{y+z+2}{(y+1)(z+1)}\le\frac{x}{x+1}$$
$i.e.,$ $xy + xz + 2x + y + z + 2 \le xyz + xy + xz + x$
$i.e.,$ $x + y + z + 2 \le xyz$
$i.e.,$ $2(a + b + c) + 2 \le (a + b)(b + c)(c + a)$
$i.e.,$ $2(a + b + c) \le a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2$.
By the AM-GM inequality,
$$(a^2b + a^2c + 1) \ge 3\sqrt[3]{a^4bc} = 3a$$
Likewise, $(b^2c + b^2a + 1) \ge 3b$
and $(c^2a + c^2b + 1) \ge 3c$.
Therefore we only need to prove that
$2(a + b + c) + 3 \le 3(a + b + c)$
$i.e.,$ $3 \le a + b + c$,
which is evident from AM-GM inequality and $abc = 1$.

2nd solution : Let $a = a_1{}^3$, $b = b_1{}^3$, $c = c_1{}^3$.
Then $a_1b_1c_1 = 1$.
Note that $a_1^3 + b_1^3 + a_1^2 b_1 - a_1b_1^2$
$$= (a_1 - b_1)(a_1^2 - b_1^2) \ge 0,$$
which implies that
$a_1{}^3 + b_1{}^3 \ge a_1b_1(a_1 + b_1)$.
Therefore,
$$\frac{1}{a+b+1}=\frac{1}{a_1^3+b_1^3+a_1b_1c_1}\le\frac{1}{a_1b_1(a_1+b_1)+a_1b_1c_1}$$
$$=\frac{a_1b_1c_1}{a_1b_1(a_1+b_1+c_1)}=\frac{c_1}{a_1+b_1+c_1}$$

Likewise, $\dfrac{1}{b+c+1} \le \dfrac{a_1}{a_1+b_1+c_1}$

and $\dfrac{1}{c+a+1} \le \dfrac{b_1}{a_1+b_1+c_1}$

Adding the three inequalities yields the desired result.

2. The statement is true if and only if $k \ge 4$.

We start by proving that it does hold for each $k \ge 4$.

Consider any polynomial $F(c)$ with integer coefficients satisfying the inequality $0 \le F(c) \le k$ for each $c \in \{0, 1, \ldots, k + 1\}$.

Note first that $F(k + 1) = F(0)$, since $F(k + 1) - F(0)$ is a multiple of $k + 1$ not exceeding k in absolute value.

Hence, $F(x) - F(0) = x(x - k - 1)\, G(x)$,

where $G(x)$ is a polynomial with integer coefficients.

Consequently,

$k \ge |F(c) - F(0)| = c(k + 1 - c)|\, G(c)|$ ...(i)

for each $c \in \{1, 2, \ldots, k\}$.

The equality $c(k + 1 - c) > k$ holds for each $c \in \{2, 3, \ldots, k - 1\}$, as it is equivalent to $(c - 1)(k - c) > 0$.

Note that the set $\{2, 3, \ldots, k - 1\}$ is not empty if $k \ge 3$, and for any c in this set, (i) implies that $|G(c)| < 1$.

Since $G(c)$ is an integer, $G(c) = 0$.

Thus, $F(x) - F(0) = x(x - 2)(x - 3) \ldots (x - k + 1)$ $(x - k - 1)\, H(x)$, ...(ii)

where $H(x)$ is a polynomial with integer coefficients.

To complete the proof of our claim, it remains to show that $H(1) = H(k) = 0$.

Note that for $c = 1$ and $c = k$, (2) implies that

$k \ge |F(c) - F(0)| = (k - 2)!\cdot k \cdot |H(c)|$.

For $k \ge 4$, $(k - 2)! > 1$.

Hence $H(c) = 0$.

We established that the statement in the question holds for any $k \ge 4$.

But the proof also provides information for the smaller values of k as well.

More exactly, if $F(x)$ satisfies the given condition then 0 and $k + 1$ are roots of $F(x)$ and $F(0)$ for any $k \ge 1$; and if $k \ge 3$ then 2 must also be a root of $F(x) - F(0)$.

Taking this into account, it is not hard to find the following counter examples:

$F(x) = x\,(2 - x)$ for $k = 1$,

$F(x) = x(3 - x)$ for $k = 2$,

$F(x) = x(4 - x)\,(x - 2)^2$ for $k = 3$.

3. **1st solution :** Let p and q be the roots of $x^2 - x - 1 = 0$.

By Vieta's theorem, $p + q = 1$ and $pq = -1$.

Note that p and q must also be the roots of $ax^{17} + bx^{16} + 1 = 0$. Thus

$ap^{17} + bp^{16} = -1$ and $aq^{17} + bq^{16} = -1$.

Multiplying the first of these equations by q^{16}, the second one by p^{16}, and using the fact that $pq = -1$, we find

$ap + b = -q^{16}$ and $aq + b = -p^{16}$. ...(i)

Thus, $a = \dfrac{p^{16} - q^{16}}{p - q}$

$= (p^8 + q^8)(p^4 + q^4)(p^2 + q^2)(p + q)$

Since, $p + q = 1$,

$p^2 + q^2 = (p + q)^2 - 2pq = 1 + 2 = 3$,

$p^4 + q^4 = (p^2 + q^2)^2 - 2p^2q^2 = 9 - 2 = 7$,

$p^8 + q^8 = (p^4 + q^4)^2 - 2p^4q^4 = 49 - 2 = 47$,

it follows that $a = 1\cdot 3\cdot 7\cdot 47 = 987$.

Likewise, eliminating a in (i) gives $-b = \dfrac{p^{17} - q^{17}}{p - q}$

$= p^{16} + p^{15}q + p^{14}q^2 + \ldots + q^{16}$

$= (p^{16} + q^{16}) + pq(p^{14} + q^{14}) + p^2q^2(p^{12} + q^{12}) + \ldots$
$+ p^7q^7(p^2 + q^2) + p^8q^8$

$= (p^{16} + q^{16}) - (p^{14} + q^{14}) + \ldots - (p^2 + q^2) + 1$.

For $n \ge 1$, let $k_{2n} = p^{2n} + q^{2n}$.

Then $k_2 = 3$ and $k_4 = 7$, and

$k_{2n+4} = p^{2n+4} + q^{2n+4}$

$= (p^{2n+2} + q^{2n+2})(p^2 + q^2) - p^2q^2(p^{2n} + q^{2n})$

$= 3k_{2n+2} - k_{2n}$

For $n \ge 3$. Then $k_6 = 18$, $k_8 = 47$, $k_{10} = 123$, $k_{12} = 322$, $k_{14} = 843$, $k_{16} = 2207$.

Hence, $-b = 2207 - 843 + 322 - 123 + 47 - 18$
$+ 7 - 3 + 1 = 1597$

or $(a, b) = (987, -1597)$

2nd solution : The other factor is of degree 15 and we write

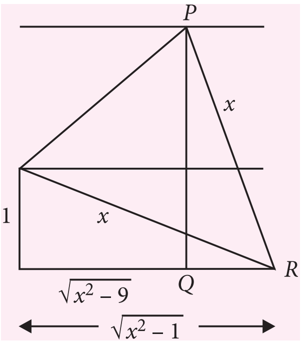
$(c_{15}x^{15} - c_{14}x^{14} + \ldots + c_1x - c_0)(x^2 - x - 1)$

$= ax^{17} + bx^{16} + 1$.

Comparing coefficients

$x^0 : c_0 = 1,$
$x^1 : c_0 - c_1 = 0, c_1 = 1$
$x^2 = - c_0 - c_1 + c_2 = 0, c_2 = 2,$
and for $3 \le k \le 15$,
$x^k : -c_{k-2} - c_{k-1} + c_k = 0.$
It follows that for $k \le 15$, $c_k = F_{k+1}$
(the Fibonacci number).
Thus $a = c_{15} = F_{16} = 987$
and $b = -c_{14} - c_{15} = -F_{17} = -1597$
or $(a, b) = (987, -1597)$.
Comment : Combining the two methods, we obtain some interesting facts about sequences k_{2n} and F_{2n-1}.
Since $3F_{2n+3} - F_{2n+5} = 2F_{2n+3} - F_{2n+4}$
$= F_{2n+3} - F_{2n+2} = F_{2n+1}$,
it follows that F_{2n-1} and k_{2n} satisfy the same recursive relation. It is easy to check that
$k_2 = F_1 + F_3$ and $k_4 = F_3 + F_5$.
Therefore, $k_{2n} = F_{2n-1} + F_{2n+1}$
and $F_{2n+1} = k_{2n} - k_{2n-2} + k_{2n-4} - \ldots$
$\ldots + (-1)^{n-1}k_2 + (-1)^n$.

4. **1st solution :** Using Pythagoras' Theorem on the triangle marked PQR we have



$$4^2 + (\sqrt{x^2-1} - \sqrt{x^2-9})^2 = x^2,$$

$$i.e.,\ 16 + x^2 - 1 + x^2 - 9 - 2\sqrt{(x^2-1)(x^2-9)} = x^2$$

This rearranges to give

$$x^2 + 6 = 2\sqrt{(x^2-1)(x^2-9)},$$

which, on squaring both sides, becomes
$x^4 + 12x^2 + 36 = 4(x^4 - 10x^2 + 9) = 4x^4 - 40x^2 + 36.$

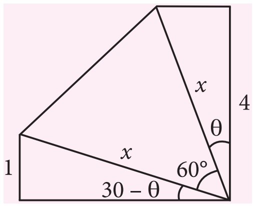
This simplifies to $3x^4 - 52x^2 = 0$, or $x^2 = \frac{52}{3}$.

The area of an equilateral triangle of side length x is half base times height,

$$i.e.,\ \frac{1}{2} \times x \times \frac{\sqrt{3}x}{2} \text{ or } x^2 \times \frac{\sqrt{3}}{4}.$$

The area of this triangle is thus $\frac{52\sqrt{3}}{12} = \frac{13\sqrt{3}}{3}$.

2nd solution : Let the side lengths of the equilateral triangle be x and the angle θ be as marked in the diagram.



First we not that $\cos\theta = \frac{4}{x}$ while $\sin\theta = \sqrt{1 - \frac{16}{x^2}}$.

Further, $\sin(30 - \theta) = \frac{\cos\theta}{2} - \frac{\sqrt{3}\sin\theta}{2} = \frac{1}{x} = \frac{\cos\theta}{4}$

So $\cos\theta = 2\sqrt{3}\sin\theta$

or $\cos^2\theta = 12\sin^2\theta = 12 - 12\cos^2\theta$ giving

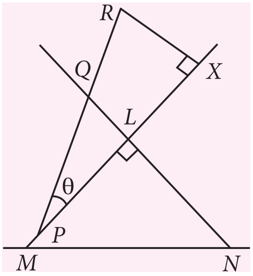
$13\cos^2\theta = 12$ or $\cos^2\theta = \frac{12}{13}$.

So, $x^2 = \frac{16}{\cos^2\theta} = \frac{(16)(13)}{12} = \frac{4(13)}{3}$

The required area is

$$\frac{1}{2}x^2 \sin 60° = \frac{\sqrt{3}}{4}x^2 = \frac{\sqrt{3}}{4} \cdot 4 \cdot \frac{13}{3} = \frac{13\sqrt{3}}{3},$$

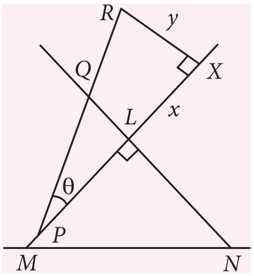
5. **1st solution :** Draw a perpendicular from R to ML. Let $\angle QPL$ be θ. Now, as MX and XR are both 45° to MN, the height of R above MN is $\frac{1}{\sqrt{2}}(MX + RX)$.



Now $LX = \cos\theta$ and $RX = 2\sin\theta$, and the height of R above MN is $\frac{1}{\sqrt{2}}(1+\cos\theta+2\sin\theta)$.

Now, since the maximum value of the expression $a\cos\theta + b\sin\theta$ is $\sqrt{a^2+b^2}$, the maximum value of $\cos\theta + 2\sin\theta$ is $\sqrt{5}$. Substituting this value in the expression above, we get the maximum value height of R above MN to be $\frac{1}{\sqrt{2}}(1+\sqrt{5}) = \frac{\sqrt{2}}{2}(1+\sqrt{5})$.

2nd solution : As before, let RX be the perpendicular from R to ML. Let x be the length of LX and y be the length of RX. Noting that $PX = 2x$ and $PR = 2$, then we have that $4x^2 + y^2 = 4$.

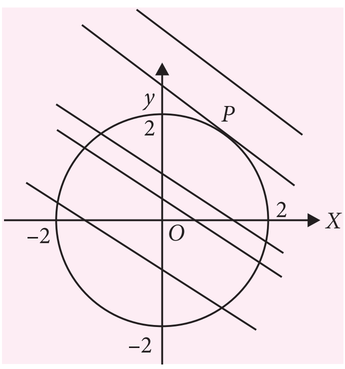


Also, the height of R above MN is $\frac{1}{\sqrt{2}}(1+x+y)$

So, we want to find the maximum value of $x + y$, given that $4x^2 + y^2 = 1$.

Replace the variable x be $X = 2x$, so that $x = \frac{X}{2}$.

Now, we want to find the maximum value of $\frac{X}{2} + y$ given that $X^2 + y^2 = 4$.



In the diagram, the circle of radius 2, centre the origin, is shown. This is the set of all points (X, y) such that $X^2 + y^2 = 4$ we want to find a point on this circle for which $\frac{X}{2} + y$ is maximum. The lines shown have slope $-\frac{1}{2}$ and are lines with constant $\frac{x}{2} + y$ — that is to say that all the points on any one of these lines have the same value of $\frac{X}{2} + y$. The value of $\frac{X}{2} + y$ increases for lines higher and to the right. From this we see that the point we are looking for is P (shown), where one of the lines is tangent to the circle. The line OP is perpendicular to this tangent and so has equation $y + 2X$. From this and the equation of the circle, we see that $X = \frac{2}{\sqrt{5}}$ and $y = \frac{4}{\sqrt{5}}$

Now, $x = \frac{X}{2} = \frac{1}{\sqrt{5}}$ and so $x + y = \frac{5}{\sqrt{5}} = \sqrt{5}$

Substituting back, the maximum height of R above MN is

$$\frac{1}{\sqrt{2}}(1+x+y) = \frac{1}{\sqrt{2}}(1+\sqrt{5}) = \frac{\sqrt{2}}{2}(1+\sqrt{5})$$

MATHS MUSING

SOLUTION SET-160

1. (c) : $N = 1 + 11 + 111 + \ldots$ to 2011 terms

$= \frac{1}{9}(9+99+999+\ldots\ldots \text{ to 2011 terms})$

$= \frac{1}{9}(10+10^2+10^3+\ldots\ldots+10^{2011}-2011)$

$= \frac{1}{81}(10^{2012}-18109)$

$= \frac{1}{81}\times 999\ldots\ldots.981891$ (begins with 2007 nines)

$= \frac{1}{9}\times 111\ldots\ldots.109099$ (begins with 2007 ones)

= 123456790 01011

$\therefore \quad S = 223(1+2+3+4+5+6+7+9)+1+1+1$

$= 223 \times 37 + 3 = 8254$

The sum of digits of S is 19.

(d) : We have, $|a+b+c| = 2$

$\Rightarrow 6 + a\bar{b} + \bar{a}b + b\bar{c} + \bar{b}c + c\bar{a} + \bar{c}a = 4$

$\Rightarrow a\bar{b} + \bar{a}b + b\bar{c} + \bar{b}c + c\bar{a} + \bar{c}a = -2$ ...(i)

$|bc + 2ca + 3ab|^2 = (bc+2ca+3ab)(\bar{b}\bar{c}+2\bar{c}\bar{a}+3\bar{a}\bar{b})$

$= 6+12+18+6\left[\Sigma(a\bar{b}+\bar{a}b)\right] = 36+6(-2) = 24$ using (i)

$\therefore \quad |bc + 2ca + 3ab| = 2\sqrt{6}$

3. (a,d) : If $t = \tan\theta$, the given equation becomes $t^4 - 8t^3 + 18t^2 - 8t + 1 = 0$

$x = t + \frac{1}{t} \Rightarrow x^2 - 8x + 16 = 0 \Rightarrow x = 4$

$t + \frac{1}{t} = 4 \Rightarrow t = 2 \pm \sqrt{3} = \tan\theta \quad \therefore \quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}.$

4. (b, d) : A plane not passing through the origin is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

It passes through (6, –4, 3) and (0, 4, –3)

$\therefore \quad \frac{6}{a}-\frac{4}{b}+\frac{3}{c}=1, \frac{4}{b}-\frac{3}{c}=1$ ....(i)

$\Rightarrow a = 3, a + b + c = 0 \Rightarrow b + c = -3$ ....(ii)

(i), (ii) $\Rightarrow b = -2, c = -1; b = 6, c = -9$

The planes are $\frac{x}{3}-\frac{y}{2}-z=1$ and $\frac{x}{3}+\frac{y}{6}-\frac{z}{9}=1$

i.e., $2x - 3y - 6z = 6$, $6x + 3y - 2z = 18$. The distance of the origin from the planes is $\frac{6}{7}, \frac{18}{7}.$

5. (c) : N = coefficient of x^{40} in the expansion of $(1 + x + x^2 + \ldots\ldots + x^{20})^4 = (1 - x^{21})^4(1 - x)^{-4}$

$\therefore \quad N = \binom{43}{40} - 4\binom{22}{19} = 12341 - 6160 = 6181$

$\therefore$ Sum of digits of $N = 16$

6. (c) : $x^2 - \frac{4}{5} > 0 \Rightarrow x^2 > \frac{4}{5}$ ...(i)

$\log_{1/5}\left(x^2-\frac{4}{5}\right) > 0 \Rightarrow x^2 - \frac{4}{5} < 1, x^2 < \frac{9}{5}$ ...(ii)

$\left(\frac{1}{2}\right)^{\log_3 \log_{1/5}\left(x^2-\frac{4}{5}\right)} > 1 = \left(\frac{1}{2}\right)^0$

$\therefore \quad \log_3 \log_{1/5}\left(x^2-\frac{4}{5}\right) < 0 = \log_3 1$

$\Rightarrow \log_{1/5}\left(x^2-\frac{4}{5}\right) < 1 \Rightarrow x^2 - \frac{4}{5} > \frac{1}{5} \Rightarrow x^2 > 1$ ...(iii)

From (i), (ii) and (iii) $x^2 \in \left(1, \frac{9}{5}\right), |x| \in \left(1, \frac{3}{\sqrt{5}}\right)$

7. (d) : The required number of ways of the outcomes is equal to the number of nondecreasing functions from the set {1, 2, 3, 4} to the set {1, 2, 3, 4, 5, 6}, which is $\binom{4+6-1}{4} = \binom{9}{4} = 126$. The probability $= \frac{126}{6^4} = \frac{7}{72}.$

8. (b) : The number of ways of choosing 3 numbers from the six numbers is $\binom{6}{3} = 20.$

The number of ways of the chosen 3 numbers appear in the list is the number of onto functions with domain 4 elements and codomain 3 elements is

$3^4 - \binom{3}{2}2^4 + \binom{3}{1}1^4 = 36 \therefore$ Probability $= \frac{20\times 36}{6^4} = \frac{5}{9}$

9. (8) : $a + b + c = 0 \Rightarrow a, b, c$ are the roots of $x^3 + qx + r = 0$ ...(i)

$\Sigma ab = q, abc = -r, a + b + c = 0 \Rightarrow \Sigma a^3 = 3abc \therefore abc = 1,$ $r = -1, \Sigma a^2 = -2q$

From (i), we get $\Sigma a^5 + q\Sigma a^3 + r\Sigma a^2 = 0$

$\Rightarrow 10 + 3q + 2q = 0 \Rightarrow q = -2$, (i) $\Rightarrow x^3 - 2x - 1 = 0.$

$\therefore \quad \Sigma a^3 - 2\Sigma a - \Sigma 1 = 0 \Rightarrow \Sigma a^4 = 2\Sigma a^2 + \Sigma a$

$\Sigma a^4 = 2(-2q) + 0 = (-4) \times (-2) = 8.$

10. (a) $\to$ (s); (b) $\to$ (q); (c) $\to$ (r)

(a) $\binom{n}{r} \div \binom{n}{r-1} = \frac{330}{165} \Rightarrow 3r = n + 1$ ...(i)

$\binom{n}{r+1} \div \binom{n}{r} = \frac{462}{330} \Rightarrow 12r = 5n - 7$ ...(ii)

From (i) & (ii), we get $n = 11, r = 4$

(b) $n = x(x + 1) = 1 \times 2, 2 \times 3, 3 \times 4, 4 \times 5, 5 \times 6, 6 \times 7, 7 \times 8, 8 \times 9, 9 \times 10$ *i.e.*, 9 values

(c) PQ is a focal chord of $y^2 = 20x$

$\Rightarrow P(5t^2, 10t), Q\left(\frac{5}{t^2}, -\frac{10}{t}\right)$

If (x, y) is the centre of the circle described on PQ as diameter, then $\frac{2x}{5} = t^2 + \frac{1}{t^2}, \frac{y}{5} = t - \frac{1}{t}$

Eliminating t, we get $\frac{2x}{5} = \frac{y^2}{25} + 2 \Rightarrow y^2 = 10(x-5)$, a parabola of latus rectum 10.

WB JEE MOCK TEST PAPER

CATEGORY-I

For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.

1. For what values of a and b the system of equation
$2x + ay + 6z = 8$
$x + 2y + bz = 5$
$x + y + 3z = 4$, has a unique solution?
(a) $a = 2, b = 3$ (b) $a \neq 2, b \neq 3$
(c) $a = -2, b = -3$ (d) none of these

2. If $\lim_{x\to a} [f(x)\cdot g(x)]$ exists, then
(a) $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist.
(b) $\lim_{x\to a} g(x)$ exists but $\lim_{x\to a} f(x)$ does not exist.
(c) both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ may exist.
(d) none of these

3. The equation of the ellipse centred at (1, 2), one focus (6, 2) and passing through the point (4, 6) is
(a) $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$
(b) $\frac{(x+1)^2}{45} + \frac{(y+2)^2}{20} = 1$
(c) $\frac{(x-1)^2}{20} + \frac{(y-2)^2}{45} = 1$
(d) $\frac{(x+1)^2}{20} + \frac{(y+2)^2}{45} = 1$

4. The A.M. between m and n and the G.M. between a and b are each equal to $\frac{ma+nb}{m+n}$. Then $m =$
(a) $\frac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ (b) $\frac{b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$
(c) $\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ (d) $\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$

5. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{2}$ then the angle between $\vec{a}$ and $\vec{b}$ is
(a) $\frac{8\pi}{2}$ (b) π (c) $\frac{\pi}{2}$ (d) none of these

6. The smallest positive integral value of 'n' such that $\left(\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right)^n$ is purely imaginary, is
(a) 2 (b) 8 (c) 4 (d) 3

7. Which term is independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$?
(a) 6 (b) 8 (c) 5 (d) 7

8. If A and B are two independent events. The probability that both A and B occurs is $\frac{1}{12}$ and probability neither A nor B occurs is $\frac{1}{2}$ then
(a) $P(A) = \frac{1}{3}, P(B) = \frac{1}{4}$
(b) $P(A) = \frac{1}{2}, P(B) = \frac{1}{6}$
(c) $P(A) = \frac{1}{6}, P(B) = \frac{1}{2}$
(d) $P(A) = \frac{1}{4}, P(B) = \frac{1}{2}$

9. $\int_{-1}^{2} \frac{|x|}{x} dx =$

(a) 1 (b) 3
(c) not exists (d) none of these

10. Ratio in which the join of (2, 1) and (–1, 2) is divided by the line $x + 3y + 5 = 0$ is
(a) 1 : 1 internally (b) 1 : 1 externally
(c) 2 : 1 externally (d) lines do not intersect

11. The length of the common chord of the parabolas $y^2 = x$ and $x^2 = y$ is
(a) $2\sqrt{2}$ (b) 1 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

12. The set of values of p for which the roots of the equation $3x^2 + 2x + p(p - 1) = 0$ are of opposite signs is
(a) $(-\infty, 0)$ (b) $(0, 1)$
(c) $(1, \infty)$ (d) $(0, \infty)$.

13. In a survey it is to be found that 70% of employees like bananas and 64% like apples. If x% like both bananas and apples, then
(a) $x \geq 34$ (b) $x \leq 64$
(c) $34 \leq x \leq 64$ (d) all of these

14. The values of a, for which the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $a\hat{i} - 3\hat{j} + \hat{k}$ respectively are the vertices of a triangle, right angled at C are
(a) 2 and 1 (b) –2 and –1
(c) –2 and 1 (d) 2 and –1

15. If ${}^nC_r + 3\,{}^nC_{r+1} + 3\,{}^nC_{r+2} + {}^nC_{r+3} = {}^{15}C_9$, then which of the following is not true ?
(a) $n = 12$ (b) $r = 8$
(c) $r = 13$ (d) both (b) and (c)

16. The straight lines $x + y = 0$, $3x + y = 4$, $x + 3y - 4 = 0$ form a triangle which is
(a) isosceles (b) equilateral
(c) right angled (d) none of these

17. A single letter is selected at random from the word "FAVOURABLE". The probability that it is a vowel is

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) none of these

18. If $C_0, C_1, C_2, ..., C_n$ denote the binomial coefficients in the expansion of $(1 + x)^n$, then

$\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + ... + n\frac{C_n}{C_{n-1}}$ equals

(a) $\frac{n}{2}$ (b) $\frac{n+1}{2}$

(c) $\frac{n(n-1)}{2}$ (d) $\frac{n(n+1)}{2}$

19. The general solution of the differential equation $\sqrt{1 - x^2y^2} \cdot dx = y \cdot dx + x \cdot dy$ is
(a) $\sin(xy) = x + c$ (b) $\sin^{-1}(xy) + x = c$
(c) $\sin(x + c) = xy$ (d) $\sin(xy) + x = c$

20. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, then $a^a \times b^b \times c^c =$
(a) 1 (b) 2 (c) –1 (d) none of these

21. If $\log\left(\frac{x+y}{5}\right) = \frac{1}{2}(\log x + \log y)$, then $\frac{x}{y} + \frac{y}{x} =$
(a) 5 (b) 10 (c) 23 (d) 1

22. Coordinates of the points A, B, C, D are (13, 7), (–5, 2), (7, 3) and (3, 7) respectively. Let AB and CD meet at P, then $PA : PB$ is equal to
(a) 6 : 7 (b) –6 : 7 (c) 7 : 6 (d) none of these

23. If $n(A) = m$, then number of relations in A are
(a) 2^m (b) $2^m - 2$
(c) 2^{m^2} (d) none of these

24. The derivative of $f(x) = \begin{cases} \frac{x-1}{2x^2 - 7x + 5} & \text{when } x \neq 1 \\ -\frac{1}{3} & \text{when } x = 1 \end{cases}$

at $x = 1$ is
(a) $-\frac{1}{9}$ (b) $-\frac{2}{9}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$

25. A line AB in three-dimensional space makes angles 45° and 120° with the positive x-axis and the positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals
(a) 30° (b) 45° (c) 60° (d) 75°

26. The function $f(x) = \frac{x}{\log x}$ increases on the interval
(a) $(0, \infty)$ (b) $(0, e)$ (c) (e, ∞) (d) none of these

27. The remainder obtained when 1! + 2! + 3! +...+ 200! divided by 14 is
(a) 3 (b) 4 (c) 5 (d) none of these

28. The sum to 50 terms of

$$\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\ldots \text{ is}$$

(a) $\frac{50}{17}$ (b) $\frac{100}{17}$ (c) $\frac{150}{17}$ (d) $\frac{200}{71}$

29. The value of

$$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$$

is

(a) $\frac{1}{2}$ (b) $\cos\frac{\pi}{8}$ (c) $\frac{1}{8}$ (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

30. $\int_{-1}^{1}(e^{x^3}+e^{-x^3})(e^x-e^{-x})\,dx$ is equal to

(a) $\frac{e^2}{2}-2e$ (b) e^2-2e

(c) $2(e^2-e)$ (d) 0

31. If roots of the equation $x^2+\alpha^2=8x+6\alpha$ are real, then which one is correct?

(a) $-2\le\alpha\le 8$ (b) $2\le\alpha\le 8$

(c) $-2<\alpha\le 8$ (d) $-2\le\alpha<8$

32. The number of values of c such that the straight line $y=4x+c$ touches the curve $\frac{x^2}{4}+y^2=1$ is

(a) 0 (b) 4 (c) 7 (d) none of these

33. If $f(x)=\frac{10^x-10^{-x}}{10^x+10^{-x}}$ is a bijective function, then find $f^{-1}(x)$.

(a) $\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$ (b) $\log_{10}(2-x)$

(c) $\frac{1}{2}\log_{10}(2x-1)$ (d) $\frac{1}{4}\log_{10}\left(\frac{2x}{2-x}\right)$

34. The distance between the lines $5x-12y+65=0$ and $5x-12y-39=0$ is

(a) 4 units (b) 16 units

(c) 2 units (d) 8 units

35. The number of solution of $\log_2(x+5)=6-x$ is

(a) 2 (b) 1 (c) 3 (d) none of these

36. The equation of the tangent to $4x^2-9y^2=36$ which is perpendicular to the straight line $5x+2y-10=0$ is

(a) $5(y-3)=4\left(x-\frac{\sqrt{11}}{2}\right)$

(b) $2x-5y+10-12\sqrt{3}=0$

(c) $2x-5y-10+12\sqrt{3}=0$

(d) no such tangent exists

37. Let $A(1,-1,2)$ and $B(2,3,-1)$ be two points. If a point P divides AB internally in the ratio 2 : 3, then the position vector of P is

(a) $\frac{1}{\sqrt{5}}(\hat{i}+\hat{j}+\hat{k})$ (b) $\frac{1}{\sqrt{3}}(\hat{i}+6\hat{j}+\hat{k})$

(c) $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$ (d) $\frac{1}{5}(7\hat{i}+3\hat{j}+4\hat{k})$

38. $\sum_{r=0}^{n-1}\frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}$ is equal to

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n+1}{n-1}$

(c) $\frac{n+1}{2}$ (d) $\frac{n}{2}$

39. If m and n denote respectively the order and degree of a differential equation

$$\left[a+\left(\frac{dy}{dx}\right)^6\right]^{7/5}=b\frac{d^2y}{dx^2}$$

then the value of (m, n) will be

(a) (1, 7) (b) (1, 6) (c) (2, 5) (d) (2, 6)

40. The solution of $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$ is

(a) -13 (b) 13 (c) $\sqrt{13}$ (d) none of these

41. If 2 events A & B are such that $P(A^c)=0.3$, $P(B)=0.4$ and $P(A\cap B^c)=0.5$ then $P(B/A\cup B^c)$ is

(a) 0.9 (b) 0.25 (c) 0.5 (d) 0.8

42. Equation of the plane through $(-1,-1,1)$ which is parallel to $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=0$ is

(a) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})+1=0$

(b) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-1=0$

(c) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})+3=0$

(d) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-3=0$

43. The number of ways of selecting 10 balls out of an unlimited number of white, red, blue and green ball is

(a) 270 (b) 280 (c) 286 (d) 90

44. The coordinates of the foot of the perpendicular from (0, 0) upon the line $x + y = 2$ are

(a) (2, –1) (b) (–2, 1) (c) (1, 1) (d) (1, 2)

45. The eccentricity of the conic $3x^2 + 4y^2 - 6x - 8y + 4 = 0$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) none of these

46. If $f(x) = \frac{10\cos x + 5\cos 3x + \cos 5x}{\cos 6x + 6\cos 4x + 15\cos 2x + 10}$, then $f(0) + f'(0) + f''(0) =$

(a) 0 (b) 1 (c) 1/2 (d) – (1/2)

47. Let $P(n) = n\,(n + 1)$ is an even number, then which of the following is true?

(a) $P(3)$ (b) $P(100)$ (c) $P(50)$ (d) all of these

48. If a line makes angles α, β, γ with x, y, and z axis, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma =$

(a) 1 (b) 2 (c) –1 (d) –3/2

49. The range of the function $f(x) = 9^x - 3^x + 1$ is

(a) $(-\infty, \infty)$ (b) $(-\infty, 0)$

(c) $(0, \infty)$ (d) $\left[\frac{3}{4}, \infty\right)$

50. If $f(x) = \begin{cases} \frac{(1-\sin^3 x)}{3\cos^2 x}, & \text{when } x < \frac{\pi}{2} \\ a, & \text{when } x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi - 2x)^2}, & \text{when } x > \frac{\pi}{2} \end{cases}$

is continuous at $x = \frac{\pi}{2}$. Then (a, b) is

(a) $\left(\frac{1}{2}, 4\right)$ (b) $\left(1, \frac{1}{4}\right)$

(c) $\left(2, \frac{1}{4}\right)$ (d) none of these

51. The value of $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right) + 3\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) - 4\tan^{-1}(-1)$ is equal to

(a) $\frac{7\pi}{4}$ (b) $\frac{11\pi}{4}$ (c) $\frac{\pi}{12}$ (d) $\frac{13\pi}{4}$

52. If α, β are roots of the equation $x^2 - p(x + 1) - q = 0$, then the value of $\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + q} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + q}$ is

(a) 0 (b) 2 (c) 1 (d) – 1

53. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ then A^{-1} is

(a) $-\frac{1}{9}A$ (b) $\frac{1}{9}A$ (c) $\frac{1}{19}A$ (d) $-\frac{1}{19}A$

54. The vectors $\vec{a} = \hat{i} + \hat{j} + m\hat{k}$, $\vec{b} = \hat{i} + \hat{j} + (m+1)\hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + m\hat{k}$ are coplanar if m is equal to

(a) 1 (b) 4 (c) 3

(d) no value of m for which vectors are co-planar.

55. If A.M. between two numbers is 5 and their G.M. is 4, then the numbers are

(a) 2, 8 (b) 4, 4 (c) 6, 4 (d) none of these

56. If X is a binomial variate with the range {0, 1, 2, 3, 4, 5, 6} and $P\,(X = 2) = 4\,P\,(X = 4)$, then the parameter p of X is

(a) 1/3 (b) 1/2 (c) 1/4 (d) 3/4

57. The rank of the word 'FLOWER' is

(a) 165 (b) 155 (c) 145 (d) none of these

58. The length of latus rectum of the parabola

$169\,[(x - 1)^2 + (y - 3)^2] = (5x - 12y + 17)^2$ is

(a) $\frac{14}{13}$ (b) $\frac{28}{13}$ (c) $\frac{12}{13}$ (d) none of these

59. In which of the following functions, Rolle's theorem is applicable?

(a) $f(x) = |x|, -2 \le x \le 2$

(b) $f(x) = \tan x, 0 \le x \le \pi$

(c) $f(x) = 1 + (x - 2)^{2/3}, 1 \le x \le 3$

(d) $f(x) = x(x - 2)^2, 0 \le x \le 2$

60. If a, b, c, d are in G.P., then

$\frac{(a^2 + b^2 + c^2)(b^2 + c^2 + d^2)}{(ab + bc + cd)^2} =$

(a) 1 (b) 2 (c) 3 (d) none of these

CATEGORY-II

Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.

61. If $y = \log_x(\log_e x)(\log_e x)$, then $\frac{dy}{dx}$ equals

(a) $\frac{1}{x\log_x \log_x x}$ (b) $\frac{1}{x\log_e x}$

(c) 0 (d) none of these

62. For hyperbola $xy = 4$, which is not true ?

(a) equations of transverse axis is $y \pm x = 0$

(b) eccentricity, $e = \sqrt{2}$

(c) co-ordinates of foci are $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$ and equation of directrix is given by $x + y \pm 2\sqrt{2} = 0$

(d) none of these

63. Let $P(n) = \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number is a true statement

(a) only for $n > 1$

(b) only for n is an odd positive integer

(c) only for n is an even positive integer

(d) $\forall\ n \in N$.

64. The slope of the tangent at (x, y) to a curve passing through $\left(1, \frac{\pi}{4}\right)$ is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$. Then the equation of the curve is

(a) $y = \tan^{-1}\left(\log\left(\frac{e}{x}\right)\right)$

(b) $x\tan^{-1}\left(\log\left(\frac{x}{e}\right)\right)$

(c) $y = x\tan^{-1}\left(\log\left(\frac{e}{x}\right)\right)$

(d) none of these

65. In a right-angled triangle, the sides are a, b and c, with c as hypotenuse, and $c - b \neq 1$, $c + b \neq 1$. Then the value of $(\log_{c+b} a + \log_{c-b} a)/(2\log_{c+b} a \times \log_{c-b} a)$ will be

(a) 2 (b) –1 (c) $\frac{1}{2}$ (d) 1

66. The displacement y (in metres) of a body varies with time t (in seconds) as $y = \frac{-2}{3}t^2 + 16t - 12$. How long does the body take to come to rest?

(a) 16 secs (b) 12 secs

(c) 10 secs (d) 8 secs

67. The solution of the equation $(3|x| - 3)^2 = |x| + 7$ which belongs to the domain of the function $y = \sqrt{x(x-3)}$ are given by

(a) $\pm\frac{1}{9}, \pm 2$ (b) $-\frac{1}{9}, 2$

(c) $\frac{1}{9}, -2$ (d) $-\frac{1}{9}, -2$.

68. A point initially at rest moves along the x-axis. Its acceleration varies with time as $a = (5t + 6)\text{m/s}^2$. If it starts from the origin, the distance covered by it in 2 second is

(a) 18.66 m (b) 14.33 m

(c) 12.18 m (d) 6.66 m.

69. The mean deviation and S.D. about actual mean of the series a, $a + d$, $a + 2d$, ..., $a + 2nd$ are respectively

(a) $\frac{n(n+1)d}{2n+1}, \sqrt{\frac{n(n-1)}{3}}\cdot d$

(b) $\frac{n(n-1)}{3}, \frac{n(n+1)}{2n}\cdot d$

(c) $\frac{n(n+1)d}{(2n+1)}, \sqrt{\frac{n(n+1)}{3}}\cdot d$

(d) $\frac{n(n-1)d}{(2n-1)}, \sqrt{\frac{n(n-1)}{3}}\cdot d$

70. A plane which is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$, passes through $(1, -2, 1)$. The distance of the plane from the point $(1, 2, 2)$ is

(a) 0 unit (b) 1 unit (c) $\sqrt{2}$ units (d) $2\sqrt{2}$ units

CATEGORY-III

In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates marks one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without making any incorrect, formula below will be used to allot marks. 2 × (no. of correct response/total no. of correct options)

71. The equations of the circles which touch both the axes and the line $4x + 3y = 12$ and have centres in the first quadrant, are

(a) $x^2 + y^2 - x - y + 1 = 0$

(b) $x^2 + y^2 - 2x - 2y + 1 = 0$

(c) $x^2 + y^2 - 12x - 12y + 36 = 0$

(d) $x^2 + y^2 - 6x - 6y + 36 = 0$

72. If $a \le \tan^{-1}x + \cot^{-1}x + \sin^{-1}x \le b$, then

(a) $a = 0$ (b) $b = \frac{\pi}{2}$

(c) $a = \frac{\pi}{4}$ (d) $b = \pi$

73. If $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$, then $x =$

(a) 1 (b) -1 (c) $\sqrt{2}+1$ (d) $\sqrt{2}-1$

74. If A and B are two events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{8}$, then

(a) $P(A \cup B) \ge \frac{3}{4}$

(b) $P(A' \cap B) \le \frac{1}{4}$

(c) $\frac{3}{8} \le P(A \cap B) \le \frac{5}{8}$

(d) none of these

75. If the vectors $\hat{i}-\hat{j}$, $\hat{j}+\hat{k}$ and a form a triangle, then a may be

(a) $-\hat{i}-\hat{k}$ (b) $\hat{i}-2\hat{j}-\hat{k}$

(c) $2\hat{i}+\hat{j}+\hat{k}$ (d) $\hat{i}+\hat{k}$

76. The area of a triangle is 5 sq. units. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. The coordinates of the third vertex can be

(a) $\left(-\frac{3}{2}, \frac{3}{2}\right)$ (b) $\left(-\frac{3}{4}, -\frac{3}{2}\right)$

(c) $\left(\frac{7}{2}, \frac{13}{2}\right)$ (d) $\left(-\frac{1}{4}, \frac{11}{4}\right)$

77. The vector $\frac{2\hat{i}-2\hat{j}+\hat{k}}{3}$

(a) is a unit vector

(b) is parallel to $-\hat{i}+\hat{j}-\frac{1}{2}\hat{k}$

(c) is perpendicular to the vector $3\hat{i}+2\hat{j}-2\hat{k}$

(d) makes the angle $\cos^{-1}\frac{5}{\sqrt{29}}$ with the vector $2\hat{i}-4\hat{j}+3\hat{k}$

78. If $A = \begin{bmatrix} \sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then which of the following are true?

(a) $|A^T| = 1$ (b) $|A^{-1}| = 1$

(c) $A^{-1} = \text{adj } A$ (d) $|AA^T| = 10$

79. $\int \frac{dx}{x^2(x^4+1)^{3/4}} = A\left(\frac{x^4+1}{x^4}\right)^B + C$

(a) $A = -1$ (b) $B = \frac{1}{4}$

(c) $A = \frac{1}{2}$ (d) $B = \frac{1}{2}$

80. If $u(x)$ and $v(x)$ are two independent solutions of the differential equation $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, then additional solution(s) of the given differential equation is(are)

(a) $y = c_1\, u(x) + c_2\, v(x)$, c_1 and c_2 are arbitrary constants

(b) $y = c_1\{u(x) - v(x)\} + c_2\, v(x)$, c_1 and c_2 are arbitrary constants

(c) $y = c_1\, u(x)\, v(x) + c_2\, u(x)/v(x)$, c_1 and c_2 are arbitrary constants

(d) $y = u(x)\, v(x)$

ANSWER KEY

1. (b)	**2.** (c)	**3.** (a)	**4.** (d)	**5.** (d)
6. (b)	**7.** (d)	**8.** (a)	**9.** (a)	**10** (d)
11. (c)	**12.** (b)	**13.** (d)	**14.** (a)	**15.** (d)
16. (a)	**17.** (b)	**18.** (d)	**19.** (c)	**20.** (a)
21. (c)	**22.** (d)	**23.** (c)	**24.** (b)	**25.** (c)
26. (c)	**27.** (c)	**28.** (b)	**29.** (c)	**30.** (d)
31. (a)	**32.** (d)	**33.** (a)	**34.** (d)	**35.** (b)
36. (d)	**37.** (d)	**38.** (d)	**39.** (c)	**40.** (b)
41. (b)	**42.** (a)	**43.** (c)	**44.** (c)	**45.** (a)
46. (b)	**47.** (d)	**48.** (c)	**49.** (d)	**50.** (a)
51. (d)	**52.** (c)	**53.** (c)	**54.** (d)	**55.** (a)
56. (a)	**57.** (b)	**58.** (b)	**59.** (d)	**60.** (a)
61. (b)	**62** (d)	**63.** (d)	**64.** (c)	**65.** (d)
66. (b)	**67.** (d)	**68.** (a)	**69.** (c)	**70.** (d)
71. (b, c)	**72.** (a, d)	**73.** (a, b, c d)	**74.** (a, b, c)	
75. (a,b, d)	**76.** (a, c)	**77.** (a,b,c, d)	**78.** (a, b, c)	
79. (a,b)	**80.** (a,b)			

JEE ADVANCED PRACTICE PAPER

READY STEADY GO!!

EXAM ON 22ND MAY

PAPER-1

SECTION-1

INTEGER ANSWER TYPE

1. If $(1+x)^n = \sum_{r=1}^{n} C_r x^r$, then the value of $C_1 + 2C_2 + 3C_3 + ... + nC_n$ equals $n \cdot k^{n-1}$. Find k

2. Let $f: (0, \infty) \rightarrow R$ and $F(x) = \int_0^x f(t)dt$. If $F(x^2) = x^2(1 + x)$, then $f(4)$ equals

3. The number of points in $(-\infty, \infty)$, for which $x^2 - x \sin x - \cos x = 0$, is

4. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then $n =$

5. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is

6. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010(x - 2009)(x - 2010)^2(x - 2011)^3(x - 2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is

7. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is

8. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. The number of distinct complex numbers z satisfying
$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$ is equal to

SECTION-2

ONE OR MORE THAN ONE CORRECT ANSWER TYPE

9. A line l passing through the origin is perpendicular to the lines
$l_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}, \ -\infty < t < \infty$
$l_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}, \ -\infty < s < \infty$
Then the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is(are)
(a) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (b) $(-1, -1, 0)$
(c) $(1, 1, 1)$ (d) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

10. Let $f: [a, b] \rightarrow [1, \infty)$ be a continuous function and let $g : R \rightarrow R$ be defined as
$$g(x) = \begin{cases} 0 \quad , & \text{if} \quad x < a \\ \int_a^x f(t)dt, & \text{if} \quad a \le x \le b \\ \int_a^b f(t)dt, & \text{if} \quad x > b \end{cases}$$
Then
(a) $g(x)$ is continuous but not differentiable at a
(b) $g(x)$ is differentiable on R
(c) $g(x)$ is continuous but not differentiable at b
(d) $g(x)$ is continuous and differentiable at either a or b but not both

11. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(ω) denotes the principal argument of a non-zero complex number ω, then
(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(b) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

(c) $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix}=0$

(d) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

12. Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

(a) the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$

(b) a focus of the hyperbola is (2, 0)

(c) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(d) the equation of the hyperbola is $x^2 - 3y^2 = 3$

13. If $f(x) = \min\{1, x^2, x^3\}$, then

(a) $f(x)$ is continuous $\forall\ x \in R$

(b) $f'(x) > 0,\ \forall\ x > 1$

(c) $f(x)$ is differentiable but continuous $\forall\ x \in R$

(d) $f(x)$ is not differentiable for two values of x.

14. Which of the following functions are continuous on $(0, \pi)$?

(a) $\tan x$ (b) $\int_0^x t\sin(1/t)dt$

(c) $\begin{cases} 1 & , \ 0 < x \le \frac{3\pi}{4} \\ 2\sin\frac{2}{9}x & , \ \frac{3\pi}{4} < x < \pi \end{cases}$

(d) $\begin{cases} x\sin x & , \ 0 < x \le \frac{\pi}{2} \\ \frac{1}{2}\pi\sin(\pi+x) & , \ \frac{\pi}{2} < x < \pi \end{cases}$

15. If $\cos x - \sin\alpha\cot\beta\sin x = \cos\alpha$, then $\tan(x/2)$ is equal to

(a) $\cot\frac{1}{2}\alpha\tan\frac{1}{2}\beta$ (b) $-\cot\frac{1}{2}\beta\tan\frac{1}{2}\alpha$

(c) $\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta$ (d) $\cot\frac{1}{2}\alpha\cot\frac{1}{2}\beta$

16. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ with its sides parallel to the axes of coordinates. The coordinates of the vertices are

(a) $(-6,-9)$ (b) $(-6, 5)$ (c) $(8, -9)$ (d) $(8, 5)$

17. A tangent drawn to the curve $y = f(x)$ at $P(x, y)$ cuts the x-axis and y-axis at A and B respectively such that $BP : AP = 3 : 1$, given that $f(1) = 1$, then

(a) equation of curve is $x\frac{dy}{dx}-3y=0$

(b) normal at (1, 1) is $x + 3y = 4$

(c) curve passes through (2, 1/8)

(d) equation of curve is $x\frac{dy}{dx}+3y=0$

18. If the first and $(2n - 1)^{\text{th}}$ terms of an A.P., a G.P. and H.P. are equal and their n^{th} terms are a, b and c respectively, then

(a) $a = b = c$ (b) $a + c = b$

(c) $a \ge b \ge c$ (d) $ac - b^2 = 0$.

SECTION-3

MATRIX MATCH TYPE

19. Match the statements/expressions given in Column I with the values given in Column II.

	Column I		Column II
(A)	Root(s) of the equation $2\sin^2\theta + \sin^2 2\theta = 2$	(p)	$\pi/6$
(B)	Points of discontinuity of the function $f(x)=\left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$ where $[y]$ denotes the largest integer less than or equal to y	(q)	$\pi/4$
(C)	Volume of the parallelepiped with its edges represented by the vectors $\hat{i}+\hat{j},\ \hat{i}+2\hat{j}$ and $\hat{i}+\hat{j}+\pi\hat{k}$ is	(r)	$\pi/3$
(D)	Angle between vectors $\vec{a}$ and $\vec{b}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a}+\vec{b}+\sqrt{3}\vec{c}=\vec{0}$	(s)	$\pi/2$
		(t)	π

20. Match the following .

	Column I		Column II
(A)	The minimum value of $\frac{x^2+2x+4}{x+2}$ is	(p)	0
(B)	Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible values of k are	(q)	1

(C)	Let $a = \log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	(r)	2
(D)	If $\sin\theta = \cos\phi$, then the possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ are	(s)	3

PAPER-2

SECTION-1

INTEGER ANSWER TYPE

1. Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is

2. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to

3. The maximum value of the expression

$$\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta} \text{ is } \_\_\_\_\_$$

4. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let N be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through N is also a common tangent to C_2 and C, then the radius of the circle C is

5. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \leq -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is

6. Let $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{i} - \hat{j}$ and $\vec{w} = \hat{i} + 2\hat{j} + 3\hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{u}\cdot\hat{n} = 0$ and $\vec{v}\cdot\hat{n} = 0$, then $|\vec{w}\cdot\hat{n}| =$

7. Two persons A and B have equal number of sons. There are three cinema tickets which are to be distributed among the sons of A and B. The probability that all the tickets go to sons of A is $\frac{1}{20}$. The number of sons to each of them is

8. The vectors $2\hat{i} + 3\hat{j}$, $5\hat{i} + 6\hat{j}$ and $8\hat{i} + \lambda\hat{j}$ have their initial point at (1, 1). The value of λ so that the vectors terminate on one straight line is

SECTION-2

ONE OR MORE THAN ONE CORRECT ANSWER TYPE

9. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

is(are)

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

10. $f(x) = \begin{cases} e^x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, $x \in [1, 3]$ then $g(x)$ has

(a) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$

(b) local maxima at $x = 1$ and local minima at $x = 2$

(c) no local maxima

(d) no local minima.

11. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$, for $n = 1, 2, 3, \ldots$. Then,

(a) $S_n < \frac{\pi}{3\sqrt{3}}$ (b) $S_n > \frac{\pi}{3\sqrt{3}}$

(c) $T_n < \frac{\pi}{3\sqrt{3}}$ (d) $T_n > \frac{\pi}{3\sqrt{3}}$

12. A bag contains four tickets with numbers 112, 121, 211, 222. One ticket is drawn at random from the bag. Let E_i $(i = 1, 2, 3)$ denote the event that i^{th} digit on the drawn ticket is 2. Then

(a) E_1, E_2, E_3 are pairwise independent

(b) E_1, E_2 are independent

(c) E_2 and E_3 are not independent

(d) E_1, E_2, E_3 are mutually independent.

13. Internal bisector of $\angle A$ of triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent sides of ΔABC then
(a) AE is H.M. of b and c
(b) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$ (c) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$
(d) the triangle AEF is isosceles.

14. If a, b and c are three positive real numbers, which one of the following hold?
(a) $\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \geq a+b+c$
(b) $(b + c)(c + a)(a + b) \geq 8\,abc$
(c) $a^2 + b^2 + c^2 < ab + bc + ca$
(d) none of these

15. A value of b for which the equations $x^2 + bx - 1 = 0$, $x^2 + x + b = 0$, have one root in common is
(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$
(c) $i\sqrt{5}$ (d) $\sqrt{2}$

16. $\int \frac{1}{\sin x + \sqrt{3}\cos x}dx =$
(a) $\frac{1}{2}\log\tan\left(\frac{1}{2}x+\frac{1}{6}\pi\right)$
(b) $\frac{1}{2}\log\left[\operatorname{cosec}\left(x+\frac{\pi}{3}\right)-\cot\left(x+\frac{\pi}{3}\right)\right]$
(c) $\frac{1}{2}\log\left[\sec\left(x-\frac{\pi}{6}\right)+\tan\left(x-\frac{\pi}{6}\right)\right]$
(d) $-\frac{1}{2}\log\left[\operatorname{cosec}\left(x+\frac{\pi}{3}\right)+\cot\left(x+\frac{\pi}{3}\right)\right]$

SECTION-3

COMPREHENSION TYPE

PARAGRAPH -1

Given that for each $a \in (0, 1)$, $\lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$.

17. The value of $g\left(\frac{1}{2}\right)$ is
(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

18. The value of $g'\left(\frac{1}{2}\right)$ is
(a) $\frac{\pi}{2}$ (b) π (c) $-\frac{\pi}{2}$ (d) 0

PARAGRAPH -2

Consider the polynomial:
$f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

19. The real numbers lies in the interval
(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$
(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

20. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval
(a) $\left(\frac{3}{4}, 3\right)$ (b) $\left(\frac{21}{64}, \frac{11}{16}\right)$
(c) $(9, 10)$ (d) $\left(0, \frac{21}{64}\right)$

SOLUTIONS

PAPER-1

1. **(2) :** $C_1 + 2C_2 + 3C_3 + ... + nC_n$
$= n + 2.\frac{n(n-1)}{2!} + \frac{3n(n-1)(n-2)}{3!} + ... + n$
$= n\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + ... + 1\right]$
$= n(1 + 1)^{n-1} = n2^{n-1} \Rightarrow k = 2$

2. **(4) :** $F(x) = \int_0^x f(t)dt \Rightarrow F(x^2) = \int_0^{x^2} f(t)dt = x^2(1+x)$
$\Rightarrow f(x^2)\cdot 2x = 2x + 3x^2 \Rightarrow f(4) = 4.$

3. **(2)** 4. **(6)** 5. **(6)**

6. **(1) :** $f(x) = \ln\{g(x)\} \Rightarrow g(x) = e^{f(x)}$
$g'(x) = e^{f(x)}f'(x)$ as $e^{f(x)} \neq 0, f'(x) = 0$
Thus the point where f' change sign from + to –, we have a point of maxima

+ 2009 – 2010 – 2011 + 2012 +

7. **(2) :** $f(x) = x^3 + e^{x/2}$, we have $g(f(x)) = x$
On differentiating $g'(f(x))f'(x) = 1$
Put $x = 0$ in the above to get $g'(f(0))f'(0) = 1$
But $f(0) = 1$ and $f'(0) = \frac{1}{2}$
$\therefore \quad g'(1) = \frac{1}{f'(0)} = \frac{1}{1/2} = 2$

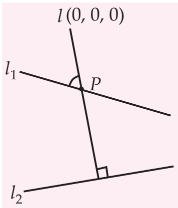
8. **(1)** $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

$\Rightarrow$ $z\{(z+\omega^2)(z+\omega) - 1 - \omega(z+\omega-1) + \omega^2(1-z-\omega^2)\} = 0$

$\Rightarrow$ Thus $z = 0$ is the only distinct solution.

9. **(b, d) :** The line l is perpendicular to l_1 and l_2. Hence the direction ratios of l are given by the vector

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -2\hat{i} + 3\hat{j} - 2\hat{k}$$



Let $P \equiv (2r, -3r, 2r)$

As it lies on l_1, we have

$$\frac{2r-3}{1} = \frac{-3r+1}{2} = \frac{2r-4}{2}$$

Thus we have $r = 1$

$\therefore$ $P \equiv (2, -3, 2)$

A point on l_2 is $(3 + 2s, 3 + 2s, 2 + s)$

Then $(3 + 2s - 2)^2 + (3 + 2s + 3)^2 + (2 + s - 2)^2 = 17$

$\Rightarrow$ $9s^2 + 28s + 20 = 0$ giving $s = -2, -10/9$

Thus the points are $(-1, -1, 0)$ and $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$.

10. **(a, c) :** $g(a^-) = 0$,

Also, $g(a^+) = \lim\limits_{h \to 0} \int\limits_a^{a+h} f(t)dt = 0$

Thus g is continuous at $x = a$. Similarly, g is continuous at $x = b$.

$$g'(a^-) = \lim_{h\to 0} \frac{g(a-h)-g(a)}{-h} = \lim_{h\to 0} \frac{0 - \int\limits_a^a f(t)dt}{-h} = \lim_{h\to 0} \frac{0}{-h} = 0$$

$$g'(a^+) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = \lim_{h\to 0} \frac{\int\limits_a^{a+h} f(t)dt - 0}{h}$$

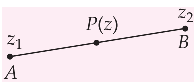
$$= \lim_{h\to 0} \frac{f(a+h)}{1} = f(a) \geq 1$$

So, $g'(a^-) \neq g'(a^+)$

Hence g is not differentiable at $x = a$.

Similarly, g is not differentiable at $x = b$.

11. **(a, c, d) :** The given statement implies that z is on the line segment joining A and B.



as $z = (1 - t)z_1 + tz_2$

Now z_1, z, z_2 are collinear,

so $\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$ giving

$\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

Note that $\text{Arg}(z - z_1) - \text{Arg}(z - z_2) = \pi$

Also $\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$ = complex slope of the line which can be rearranged as

$(z - z_1)(\bar{z}_2 - \bar{z}_1) - (z_2 - z_1)(\bar{z} - \bar{z}_1) = 0$

i.e. $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

12. **(b, d) :** Eccentricity of ellipse $= \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

Foci of ellipse are $= (\pm\sqrt{3}, 0)$

Again for the hyperbola,

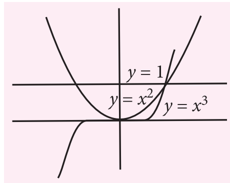
$$\sqrt{1 + \frac{b^2}{a^2}} = \frac{2}{\sqrt{3}} \Rightarrow \frac{b}{a} = \frac{1}{\sqrt{3}}$$

Also $\frac{(\sqrt{3})^2}{a^2} - 0 = 1$ $\therefore$ $a = \sqrt{3}$ which gives $b = 1$

Thus equation of hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$

Foci of hyperbola are $(\pm 2, 0)$.

13. **(a) :** $f(x) = \min\{1, x^2, x^3\}$



$$\Rightarrow f(x) = \begin{cases} x^3, & x \leq 1 \\ 1, & x > 1 \end{cases}$$

$\Rightarrow$ $f(x)$ is continuous $\forall x \in R$

and not-differentiable at $x = 1$.

14. **(b, c) :** (i) The function $\tan x$ is not continuous at $x = \pi/2 \in (0, \pi)$.

(ii) Let $F(x) = \int\limits_0^x t \sin(1/t)dt$ ...(i)

The function $f(t) = t\sin(1/t)$ is continuous for all $x \in (0, \pi)$

$\Rightarrow$ $F(x)$ is given by (i) is also continuous on $(0, \pi)$.

(iii) Let $f(x) = \begin{cases} 1, & 0 < x \leq \frac{3\pi}{4} \\ 2\sin\left(\frac{2}{9}x\right), & \frac{3\pi}{4} < x < \pi \end{cases}$

$f(x)$ is continuous at all points except possibly at $x = 3\pi/4$.

Now, $\lim\limits_{x \to \left(\frac{3\pi}{4}\right)^-} f(x) = 0$

and $\lim\limits_{x \to \left(\frac{3\pi}{4}\right)^+} = \lim\limits_{h \to 0} 2\sin\frac{2}{9}\left(\frac{3}{4}\pi + h\right) = 2\sin\frac{1}{6}\pi = 1$

$\therefore$ $f\left(\frac{3\pi}{4}^-\right) = f\left(\frac{3\pi}{4}\right) = f\left(\frac{3\pi}{4}^+\right) = 1$

$\Rightarrow$ $f(x)$ is continuous at $x = 3\pi/4$

$\Rightarrow$ $f(x)$ is continuous at all $x \in (0, \pi)$.

(iv) Let $f(x) = \begin{cases} x\sin x, & 0 < x \leq \frac{\pi}{2} \\ \frac{1}{2}\pi\sin(\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$

At $x = \pi/2$,

$$f\left(\frac{\pi}{2}^-\right) = \lim_{h\to 0}\left[\left(\frac{\pi}{2} - h\right)\sin\left(\frac{\pi}{2} - h\right)\right] = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}^+\right) = \lim_{h\to 0}\left\{\frac{1}{2}\pi\sin\left(\pi + \frac{\pi}{2} + h\right)\right\}$$

$$= \frac{1}{2}\pi \sin\left(\frac{3\pi}{2}\right) = -\frac{\pi}{2} \neq f\left(\frac{\pi}{2}^{-}\right)$$

$\Rightarrow$ $f(x)$ is not continuous at $x = \frac{\pi}{2} \in (0, \pi)$

15. (b, c) : The given equation can be written as

$$\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} - \sin\alpha\cot\beta\frac{2\tan(x/2)}{1+\tan^2(x/2)} = \cos\alpha$$

$$\Rightarrow \tan^2\frac{x}{2} + \frac{2\sin\alpha}{1+\cos\alpha}\cot\beta.\tan\frac{x}{2} - \frac{1-\cos\alpha}{1+\cos\alpha} = 0$$

$$\Rightarrow \tan^2\frac{x}{2} + 2\tan\frac{\alpha}{2}\cot\beta\tan\frac{x}{2} - \tan^2\frac{\alpha}{2} = 0$$

$$\Rightarrow \tan^2\frac{x}{2} + \frac{2\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}} \times \tan\frac{\alpha}{2}\tan\frac{x}{2} - \tan^2\frac{\alpha}{2} = 0$$

$$\Rightarrow \tan^2\frac{x}{2} + \left(\cot\frac{\beta}{2} - \tan\frac{\beta}{2}\right)\tan\frac{\alpha}{2}\tan\frac{x}{2} - \tan^2\frac{\alpha}{2} = 0$$

$$\Rightarrow \left(\tan\frac{x}{2} + \cot\frac{\beta}{2}\tan\frac{\alpha}{2}\right)\left(\tan\frac{x}{2} - \tan\frac{\beta}{2}\tan\frac{\alpha}{2}\right) = 0$$

$$\Rightarrow \tan\frac{x}{2} = -\cot\frac{\beta}{2}\tan\frac{\alpha}{2} \quad \text{or} \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

16. (a, b, c, d) : Radius of the circle $= \sqrt{(1+4+93)} = \sqrt{(98)}$

Diagonal of the square = diameter of the circle.

If a is side of the square, then $\sqrt{(a^2+a^2)} = 2\sqrt{(98)}$

$\Rightarrow$ $2a^2 = 4 \times 98 \Rightarrow a = 14$

The centre of the circle is (1, –2). Since the sides of the square are parallel to the coordinate axes, so the distance of each side from the centre is 7.

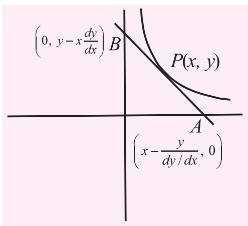
$\therefore$ Sides || to y-axis are $x = 1 + 7$ and $x = 1 - 7$,

i.e. $x = 8$ and $x = -6$, and the sides || to x-axis are

$y = -2 + 7$ and $y = -2-7$, *i.e.* $y = 5$ and $y = -9$.

Solving, the vertices of the square are (–6, –9), (–6, 5), (8, –9), and (8, 5) which is given in (a), (b) (c) and (d).

17. (a) : Equation of the tangent is



$$Y - y = \frac{dy}{dx}(X - x)$$

Given $\frac{BP}{AP} = \frac{3}{1}$ so that

$$\Rightarrow \frac{dx}{x} = \frac{dy}{3y}$$

$$\Rightarrow \ln x = \frac{1}{3}\ln y + \ln c \Rightarrow x = cy^3$$

Given $f(1) = 1 \quad \Rightarrow \quad c = 1 \quad \therefore \quad x = y^3$

18. (c, d) : Let x and y be the first term and the $(2n-1)^{\text{th}}$ terms respectively of A.P., G.P., and H.P. whose n^{th} terms are a, b, c.

$(2n-1)^{\text{th}}$ term of A.P., $y = x + (2n-2)d \Rightarrow d = \frac{(y-x)}{2(n-1)}$

$\therefore a = x + (n-1)d = \frac{1}{2}(x+y)$

$(2n-1)^{\text{th}}$ term of G.P., is

$$y = xr^{2n-2} \Rightarrow r = \left(\frac{y}{x}\right)^{\frac{1}{(2n-2)}}$$

$$\therefore \quad b = xr^{n-1} = \sqrt{xy}$$

$(2n-1)^{\text{th}}$ term of H.P. is

$$\frac{1}{y} = \frac{1}{x} + (2n-2)d_1 \Rightarrow d_1 = \frac{(x-y)}{[2xy(n-1)]}$$

$\therefore$ n^{th} term of H.P. $\frac{1}{c} = \frac{1}{x} + (n-1)d_1$

$$\Rightarrow \quad c = \frac{2xy}{(x+y)}$$

Now $a - b = \frac{1}{2}(x+y) - \sqrt{(xy)}$

$$= \frac{1}{2}\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0 \Rightarrow a \geq b$$

Now $ac = b^2$ $\therefore$ $\frac{b}{c} = \frac{a}{b} \geq 1 \Rightarrow b \geq c$

Hence $a \geq b \geq c$ and also $ac - b^2 = 0$. Note that the equality sign holds when $a = b + c$.

19. (A) → (q, s) ; (B) → (p, r, s, t) ; (C) → (t) ; (D) →(r)

(A) $2\sin^2\theta + \sin^2 2\theta = 2$

$\Rightarrow$ $2\sin^2\theta + 4\sin^2\theta\cos^2\theta = 2$

$\Rightarrow$ $\sin^2\theta + 2\sin^2\theta\cos^2\theta = 1$

$\Rightarrow$ $2\sin^2\theta\cos^2\theta - \cos^2\theta = 0$

$\Rightarrow$ $\cos^2\theta(2\sin^2\theta - 1) = 0$

$\therefore$ $\cos\theta = 0$, $\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \pi/2$, $\theta = \pi/4$ among the given choices.

(B) Let $t = \frac{3x}{\pi}$, then $f(x) = \left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$ transforms to $g(t) = [2t] \cos [t]$

Points of discontinuity of this function are

$t = \frac{1}{2}, 1, \frac{3}{2}, 3$

$\therefore$ Corresponding points are $x = \pi/6, \pi/3, \pi/2, \pi$.

(C) Volume of the parallelopiped $= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix} = \pi$.

(D) $\vec{a} + \vec{b} + \sqrt{3}\vec{c} = 0 \Rightarrow \vec{a} + \vec{b} = -\sqrt{3}\vec{c}$

$\Rightarrow |\vec{a} + \vec{b}| = \sqrt{3} \Rightarrow \sqrt{2 + 2\cos\alpha} = \sqrt{3}$, where α is the angle between $\vec{a}$ and $\vec{b}$ vectors.

$\Rightarrow$ $2 + 2\cos\alpha = 3 \Rightarrow \cos\alpha = \frac{1}{2}$ $\therefore \alpha = \frac{\pi}{3}$

20. (A) → (r); (B) → (q, s); (C) → (r, s); (D) → (p, r)

(A) Let $y = \frac{x^2 + 2x + 4}{x+2}$ $\Rightarrow x^2 + (2 - y)x + 4 - 2y = 0$

The discriminant must be non negative which yields $y^2 + 4y - 12 \geq 0$, $y \leq -6$ or $y \geq 2$

The minimum value is 2.

(B) $(A + B)(A - B) = (A - B)(A + B)$

$\Rightarrow\ A^2 - AB + BA - B^2 = A^2 + AB - BA - B^2 \Rightarrow\ AB = BA.$

Now, $(AB)^t = B^tA^t = (-B)(A) = -BA = -AB = (-1)^kAB$

Thus k has to be odd, so k can take, among the choices, the values 1 and 3.

(C) $a = \log_3(\log_3 2)$

$$\Rightarrow\ 3^{-a} = 3^{-\log_3(\log_3 2)} = 3^{\log_3 \frac{1}{\log_3 2}} = \frac{1}{\log_3 2} = \log_2 3$$

Now, $2^{-k+3^{-a}} = 2^{-k+\log_2 3} = 2^{-k}.2^{\log_2 3} = 3 \cdot 2^{-k}$

Now, $1 < 3 \cdot 2^{-k} < 2\ \Rightarrow\ \frac{1}{3} < 2^{-k} < \frac{2}{3}\ \Rightarrow\ 3 > 2^k > \frac{3}{2}$

$\Rightarrow\ \frac{3}{2} < 2^k < 3$

The only integer between 3/2 and 3 is 2, which gives $2^k = 2\ \Rightarrow\ k = 1$

Then k is less than 2 and 3.

(D) $\sin\theta = \cos\phi \Rightarrow \cos(\pi/2 \pm \theta) = \cos\phi$

$\Rightarrow\ \pi/2 - \theta = 2n\pi \pm \phi \Rightarrow\ -2n\pi = \theta \pm \phi - \pi/2$

$\Rightarrow\ \frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right) = -2n =$ an even integer.

So, it can take the values 0 or 2.

PAPER-2

1. (5)

2. (3) : $\frac{1}{2}ab\sin C = 15\sqrt{3}\ \Rightarrow\ \frac{1}{2}\times 6\times 10\sin C = 15\sqrt{3}$

$\therefore\ \sin C = \frac{\sqrt{3}}{2}\quad \therefore\ C = \frac{2\pi}{3}\quad (\because C$ is obtuse)

Now, $\cos C = \frac{a^2+b^2-c^2}{2ab}\ \Rightarrow\ -\frac{1}{2} = \frac{36+100-c^2}{2\cdot 6\cdot 10}$

$\Rightarrow 136 - c^2 = -60 \Rightarrow c^2 = 196\ \therefore\ c = 14$

$$r = \frac{\Delta}{s} = \frac{15\sqrt{3}}{\frac{6+10+14}{2}} = \frac{15\sqrt{3}}{15} = \sqrt{3}\ \therefore\ r^2 = 3$$

3. (2) : Let $f(\theta) = \sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta$

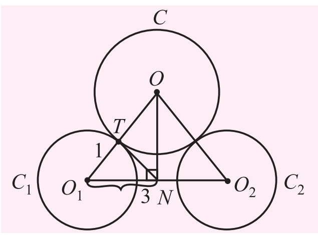
$$= \frac{1-\cos 2\theta}{2} + \frac{3\sin 2\theta}{2} + \frac{5(1+\cos 2\theta)}{2}$$

$$= \frac{6+4\cos 2\theta + 3\sin 2\theta}{2} = 3 + \frac{4\cos 2\theta + 3\sin 2\theta}{2}$$

Required value $= 3 - \frac{5}{2} = \frac{1}{2}$

Thus max. value $\frac{1}{f(\theta)} = \frac{1}{1/2} = 2$

4. (8) : $\Delta ONO_1 \sim \Delta NTO_1$



$O_1T = \frac{O_1N^2}{OO_1}$

$\Rightarrow\ 1 = \frac{9}{OO_1}$

$\therefore\ OO_1 = 9$

Hence $OT = OO_1 - O_1T = 9 - 1 = 8$

The radius of the circle C is 8.

5. (2)

6. (3) : Let $\hat{n} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\because\ \vec{u}\cdot\hat{n} = 0 \Rightarrow a_1 + a_2 = 0$ and $\vec{v}\cdot\hat{n} = 0 \Rightarrow a_1 - a_2 = 0$

$\therefore\ a_1 = 0, a_2 = 0$

Also, $a_3 = 1$ $\quad(\because \hat{n}$ is a unit vector)

$\therefore\ |\vec{w}\cdot\hat{n}| = 3.$

7. (3) : Let each have x sons.

$\therefore$ Probability that all the tickets go to the sons of A

$= \frac{{}^xC_3}{{}^{2x}C_3} = \frac{1}{20}$ (given)

$\Rightarrow\ \frac{x(x-1)(x-2)}{2x(2x-1)(2x-2)} = \frac{1}{20}\ \Rightarrow\ x = 3$

8. (9) : Since the vectors $2\hat{i}+3\hat{j}$ and $5\hat{i}+6\hat{j}$ have (1, 1) as initial point, their terminal points are (2 + 1, 3 + 1), (5 + 1, 6 + 1), *i.e.*, (3, 4) and (6, 7) respectively. The equation of line joining these points is $y-4 = \frac{7-4}{6-3}(x-3)$, *i.e.*, $x - y + 1 = 0$. The terminal point of $8\hat{i}+\lambda\hat{j}$ is $(9, \lambda + 1)$. Since the vectors terminate on the same line

$\therefore\ (9, \lambda + 1)$ lies on $x - y + 1 = 0$

$\therefore\ 9 - (\lambda + 1) + 1 = 0 \Rightarrow \lambda = 9.$

9. (c, d) : $\sum_{m=1}^{6} \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)} \cdot \frac{1}{\sin\left(\theta + \frac{m\pi}{4}\right)}$

$$= \sum_{m=1}^{6} \frac{\sin\left(\theta + \frac{m\pi}{4} - \theta - (m-1)\frac{\pi}{4}\right)}{\left\{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)\right\}} \cdot \frac{1}{\sin\frac{\pi}{4}}$$

$$= \sqrt{2}\left\{\cot\theta - \cot\left(\theta + \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) - \ldots - \cot\left(\theta + \frac{3\pi}{2}\right)\right\}$$

$$= \sqrt{2}\left\{\cot\theta - \cot\left(\theta + \frac{3\pi}{2}\right)\right\} = 4\sqrt{2}\ \text{(given)}$$

$\Rightarrow\ \cot\theta + \tan\theta = 4$ giving $\tan\theta = 2 \pm \sqrt{3}$

$\therefore\ \theta = \frac{\pi}{12}, \frac{5\pi}{12}$

10. (a, b) : $g'(x) = f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$

$g'(x) = 0 \Longrightarrow x = 1 + \ln 2$ and $x = e$

$g''(x) = \begin{cases} -e^{x-1} & 1 < x \le 2 \\ 1 & 2 < x \le 3 \end{cases}$

$g''(1 + \ln 2) = -e^{\ln 2} < 0$ hence at $x = 1 + \ln 2$, $g(x)$ has a local maximum.

$g''(e) = 1 > 0$ hence at $x = e$, $g(x)$ has local minimum

$\because f(x)$ is discontinuous at $x = 1$, then we get local maxima at $x = 1$ and local minima at $x = 2$.

11. (a, c)

12. (a, b): $P(E_1) = 2/4 = 1/2, P(E_2) = 2/4 = 1/2, P(E_3) = 1/2$

$P(E_1E_2) = 1/4, P(E_1E_3) = 1/4$

$P(E_2E_3) = 1/4, P(E_1E_2E_3) = 1/4$

$\therefore$ $P(E_1E_2) = P(E_1)\cdot P(E_2); P(E_2E_3) = P(E_2)\cdot P(E_3);$

$P(E_1E_3) = P(E_1)\cdot P(E_3)$

$\therefore$ E_1, E_2, E_3 are pairwise independent

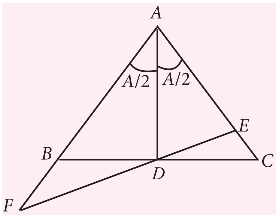
Finally, $P(E_1E_2E_3) = 1/4 \neq P(E_1)\cdot P(E_2)\cdot P(E_3)$

So, E_1, E_2, E_3 are not mutually independent.

13. (a, b, c, d) : We have $\Delta ABC = \Delta ABD + \Delta ACD$

$$\Rightarrow \frac{1}{2}bc\sin A = \frac{1}{2}c\times AD\sin\frac{A}{2}+\frac{1}{2}b\times AD\sin\frac{A}{2}$$

$$\Rightarrow AD = \frac{2bc}{b+c}\cos\frac{A}{2}$$



Again $AE = AD\sec\frac{A}{2} = \frac{2bc}{b+c}$

$\Rightarrow$ AE is H.M. of b and c.

$EF = ED + DF = 2DE$

$$= 2\times AD\tan\frac{A}{2}$$

$$= \frac{2\times 2bc}{b+c}\times\cos\frac{A}{2}\times\tan\frac{A}{2} = \frac{4bc}{b+c}\sin\frac{A}{2}$$

As $AD \perp EF$ and $DE = DF$ and AD is bisector of $\angle A$

$\Rightarrow AEF$ is isosceles triangle.

14. (a, b) : Since $a, b, c > 0$, for (a), we have to show that $a^2b^2 + b^2c^2 + c^2a^2 \geq (a + b + c)\, abc$

Now $a^2b^2 + b^2c^2 \geq 2ab^2c$ [$\because$ A.M. $\geq$ G.M.]

$b^2c^2 + c^2a^2 \geq 2abc^2, a^2b^2 + c^2a^2 \geq 2a^2bc$

Adding we get $a^2b^2 + b^2c^2 + c^2a^2 \geq abc\,(a + b + c)$

Also $a+b\geq 2\sqrt{ab}, b+c\geq 2\sqrt{(bc)}, c+a\geq 2\sqrt{(ca)}$

Multiplying we get,

$(a + b)(b + c)(c + a) \geq 8abc$

Also $a^2 + b^2 + c^2 = 1/2\,[a^2 + b^2 + b^2 + c^2 + c^2 + a^2] > ab + bc + ca$

15. (b) : $\left.\begin{array}{r} x^2+bx-1=0 \\ \text{and } x^2+x+b=0 \end{array}\right\}$ have a common root then

$$\frac{x^2}{b^2+1} = \frac{x}{-1-b} = \frac{1}{1-b}$$

Thus $x = \frac{b+1}{b-1}$ and $x^2 = \frac{b^2+1}{1-b}$

We have $\frac{(b+1)^2}{(b-1)^2} = \frac{b^2+1}{1-b}$ $\Rightarrow$ $(b^2 + 1)(1 - b) = (b + 1)^2$

$\Rightarrow$ $b^2 - b^3 + 1 - b = b^2 + 2b + 1 \Rightarrow b^3 + 3b = 0$

$\therefore$ $b = 0$ and $b^2 = -3$

A possible value of b is $-i\sqrt{3}$

We have cancelled $b = 1$ in the process but $b = 1$ is not a possible value.

16. (a, b, c, d) :

Let $I = \int\frac{1}{\sin x+\sqrt{3}\cos x}dx = \int\frac{1}{2\sin(x+\pi/3)}dx$

$$= \frac{1}{2}\int\operatorname{cosec}\left(x+\frac{\pi}{3}\right)dx = \frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)$$

which is given in (a)

or $= \frac{1}{2}\log\left[\operatorname{cosec}\left(x+\frac{\pi}{3}\right)-\cot\left(x+\frac{\pi}{3}\right)\right]$

which is given in (b)

$$= -\frac{1}{2}\log\left[\operatorname{cosec}\left(x+\frac{\pi}{3}\right)+\cot\left(x+\frac{\pi}{3}\right)\right]$$

which is given in (d)

$\because$ $\log(\operatorname{cosec}\theta - \cot\theta) = -\log(\operatorname{cosec}\theta + \cot\theta)$

Also I can be expressed as

$$I = \int\frac{dx}{2\cos\left(x-\frac{\pi}{6}\right)} = \frac{1}{2}\int\sec\left(x-\frac{\pi}{6}\right)dx$$

$$= \frac{1}{2}\log\left[\sec\left(x-\frac{\pi}{6}\right)+\tan\left(x-\frac{\pi}{6}\right)\right]$$

which is given in (c)

17. (a) : $g\left(\frac{1}{2}\right) = \lim_{h\to 0^+}\int_h^{1-h} t^{-1/2}(1-t)^{-1/2}dt$

$$= \int_0^1\frac{dt}{\sqrt{t-t^2}} = \int_0^1\frac{dt}{\sqrt{\frac{1}{4}-\left(t-\frac{1}{2}\right)^2}} = \sin^{-1}\left\{\frac{\left(t-\frac{1}{2}\right)}{\frac{1}{2}}\right\}\Bigg|_0^1$$

$= \sin^{-1}1 - \sin^{-1}(-1) = \pi$

18. (d) : We get $g(a) = g(1 - a)$ and g is differentiable.

Here $g'\left(\frac{1}{2}\right) = 0$

19. (c) : $f(x) = 1 + 2x + 3x^2 + 4x^3$

We have $f'(x) = 2 + 6x + 12x^2 = 2(6x^2 + 3x + 1)$

$D = 3^2 - 4\times 6 = -15 < 0$

Thus $f'(x)$ and so f is increasing.

$\therefore f$ has at the most one real root

Since $f\left(-\frac{1}{2}\right)f\left(-\frac{3}{4}\right) < 0$

Thus s lies on the interval $\left(-\frac{3}{4}, -\frac{1}{2}\right)$

20. (a) : As $-\frac{3}{4} < s < -\frac{1}{2}$, we have $\frac{1}{2} < t < \frac{3}{4}$

Now $\int_0^{1/2} f(x)dx < \int_0^t f(x)dx < \int_0^{3/4} f(x)dx$

we have

$$\int_0^{1/2}(4x^3+3x^2+2x+1)dx < \int_0^t f(x)dx$$

$$< \int_0^{3/4}(4x^3+3x^2+2x+1)dx$$

$$\Rightarrow [x^4+x^3+x^2+x]_0^{1/2} < \text{Area} < [x^4+x^3+x^2+x]_0^{3/4}$$

i.e. $\frac{1}{16}+\frac{1}{8}+\frac{1}{4}+\frac{1}{2} < \text{Area} < \frac{81}{256}+\frac{27}{64}+\frac{9}{16}+\frac{3}{4}$

$$\Rightarrow \frac{15}{16} < \text{Area} < \frac{525}{256}$$

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

1. How many sets of 2 and 3 (different) numbers can be formed by using numbers between 0 and 180 (both including) so that 60 is their average?

Anshul Jain, W.B

Ans. **(i) Set of 2 numbers :**

Let a and b be 2 numbers, then

$$\frac{a+b}{2}=60 \Rightarrow a+b=120$$

Let $0 \le a \le 59$ and $61 \le b \le 120$

The total no. of ways in which a can be chosen

$= {}^{60}C_1 = 60$

The value of b depends on the value of a and there is 1 value of b corresponding to 1 value of a.

$\therefore$ Total no. of sets having 2 numbers = 60.

(ii) Set of 3 numbers :

Let a, b, c be the three numbers

Then $\dfrac{a+b+c}{3}=60 \Rightarrow a+b+c=180.$

Case I: Let $0 \le a \le 59$, $0 \le b \le 59$ and $c > 60$

a can be chosen in ${}^{60}C_1 = 60$ ways

b can be chosen in ${}^{59}C_1 = 59$ ways.

($\because$ b can't use the value of a)

$\therefore$ No. of ways in which a and b can be chosen $= 60 \times 59 = 3540$.

Now $1 \le a + b \le 117$ and there is only one value of c for 1 value of $a + b$ so that $a + b + c = 180$.

$\therefore$ No. of ways in which a, b, c can be chosen $= 60 \times 59 = 3540$.

Case II: $a = 60$, $\therefore$ $b + c = 120$

The no. of ways in which b and c can assume values $= 60$ {from (i)}

$\therefore$ No. of ways in which a, b, c can be chosen $= 60$.

Case III: $61 \le a \le 90$, $61 \le b \le 90$ and $c < 60$

a can assume values in ${}^{30}C_1 = 30$ ways

b can assume values in ${}^{29}C_1 = 29$ ways

The value of c depends on the value of a and b

$\therefore$ No. of ways in which a, b, c can be chosen $= 30 \times 29 = 870$

$\therefore$ Total no. of ways in which sets of 3 numbers can be chosen $= 3540 + 60 + 870 = 4470$

$\therefore$ Total no. of ways in which sets of 2 and 3 numbers can be chosen $= 4470 + 60 = 4530$.

2. Prove that

$$2^{\left[\left(\sqrt{\log_a \sqrt[4]{ab}+\log_b \sqrt[4]{ab}}-\sqrt{\log_a \sqrt[4]{b/a}+\log_b \sqrt[4]{a/b}}\right)\cdot\sqrt{\log_a b}\right]} = \begin{cases} 2, & b \ge a > 1 \\ 2^{\log_a b}, & 1 < b < a \end{cases}$$

Karishma, Kolkata

Ans. Since,

$$\sqrt{\log_a \sqrt[4]{ab}+\log_b \sqrt[4]{ab}}$$

$$=\sqrt{\frac{1}{4}\log_a(ab)+\frac{1}{4}\log_b(ab)}$$

$$=\sqrt{\frac{1}{4}(1+\log_a b+\log_b a+1)}$$

$$=\sqrt{\left(\frac{\log_a b+\dfrac{1}{\log_a b}+2}{4}\right)}$$

$$=\sqrt{\left(\frac{\sqrt{|\log_a b|}+\dfrac{1}{\sqrt{|\log_a b|}}}{2}\right)^2}$$

$$=\frac{1}{2}\left(\sqrt{|\log_a b|}+\frac{1}{\sqrt{|\log_a b|}}\right)$$

and $\sqrt{\log_a \sqrt[4]{(b/a)}+\log_b \sqrt[4]{(a/b)}}$

$$=\sqrt{\frac{1}{4}\log_a\left(\frac{b}{a}\right)+\frac{1}{4}\log_b\left(\frac{a}{b}\right)}$$

$$=\sqrt{\frac{1}{4}(\log_a b-1+\log_b a-1)}$$

$$=\sqrt{\frac{\log_a b+\dfrac{1}{\log_a b}-2}{4}}$$

$$=\frac{\left|\sqrt{|\log_a b|}-\dfrac{1}{\sqrt{|\log_a b|}}\right|}{2}$$

Now, Let $P = \sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} -$

$\sqrt{\log_4 \sqrt[4]{b/a} + \log_b \sqrt[4]{(a/b)}}$

$$= \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} - \left|\sqrt{|\log_a b|} - \frac{1}{\sqrt{|\log_a b|}}\right|\right\}$$

Case I: If $b \geq a > 1$, then

$$P = \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} - \sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}}\right\} = \frac{1}{\sqrt{\log_a b}}$$

$\therefore \quad 2^{P\sqrt{\log_a b}} = 2^1 = 2$

Case II: If $1 < b < a$

then $P = \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} + \sqrt{|\log_a b|} - \frac{1}{\sqrt{|\log_a b|}}\right\} = \sqrt{|\log_a b|}$

$\therefore \quad 2^{P\sqrt{\log_a b}} = 2^{\log_a b}$

3. Let $\hat{a}$ be a unit vector and $\vec{b}$ be a non-zero vector not parallel to $\vec{a}$. Find the angles of the triangle two sides of which are represented by the vectors $\sqrt{3}\,(\hat{a}\times\vec{b})$ and $\vec{b}-(\hat{a}\cdot\vec{b})\,\hat{a}$.

Divyanka Choudhary, Haryana

Ans. Let $\overrightarrow{PQ} = \sqrt{3}\,(\hat{a}\times\vec{b})$

and $\overrightarrow{PR} = \vec{b} - (\hat{a}\cdot\vec{b})\,\hat{a}$

$= \hat{a}\times(\vec{b}\times\hat{a}) \qquad [\because \hat{a}\times(\vec{b}\times\hat{a}) = (\hat{a}\cdot\hat{a})\,\vec{b} - (\hat{a}\cdot\vec{b})\,\hat{a}]$

Since, $\hat{a}$ and $\vec{b}$ are non-zero vectors and $\vec{b}$ is not parallel to $\hat{a}$.

$\therefore \quad \hat{a}\times\vec{b} \neq \vec{0}$

Also, $|\hat{a}\times(\vec{b}\times\hat{a})| = |\hat{a}||\vec{b}\times\hat{a}|\sin 90°$

$[\because \hat{a}\times(\vec{b}\times\hat{a}) \perp (\vec{b}\times\hat{a})]$

$= |\vec{b}\times\hat{a}| \neq \vec{0}$

$\therefore \quad \hat{a}\times(\vec{b}\times\hat{a}) \neq \vec{0}$

Now $\overrightarrow{PQ}\cdot\overrightarrow{PR} = \sqrt{3}\,(\hat{a}\times\vec{b})\cdot[\vec{b} - (\hat{a}\cdot\vec{b})\,\hat{a}]$

$= \sqrt{3}[(\hat{a}\times\vec{b})\cdot\vec{b}] - \sqrt{3}\,(\hat{a}\cdot\vec{b})[(\vec{a}\times\vec{b})\cdot\hat{a}]$

$= 0 - 0 = 0$

$\therefore \quad \angle QPR = 90^\circ$

If $\angle PQR = \alpha$, then

$$\tan\alpha = \frac{|\overrightarrow{PR}|}{|\overrightarrow{PQ}|} = \frac{|\hat{a}\times(\vec{b}\times\hat{a})|}{\sqrt{3}\,|\hat{a}\times\vec{b}|} = \frac{|\vec{b}\times\hat{a}|}{\sqrt{3}\,|\hat{a}\times\vec{b}|}$$

$$= \frac{|\hat{a}\times\vec{b}|}{\sqrt{3}\,|\hat{a}\times\vec{b}|} = \frac{1}{\sqrt{3}} \quad \therefore \alpha = 30^\circ$$

Also, $\angle PRQ = 60°$

Thus the angles of the triangle PQR are 90°, 30° and 60°.

Students who took JEE on paper say e-test candidates at advantage

Several applicants who appeared for the offline IIT-Joint Entrance Examination (Mains) alleged on Monday that a decision to bar students from using pencil and eraser in the answer sheet benefitted online examinees.

The JEE (Mains) is held across India by the Central Board of Secondary Education (CBSE). The offline exam was held on April 3 and the online exam on April 9. But students are upset after the CBSE stopped them from carrying pen, pencil, eraser or any electronic gadget and told them to fill the Optical Mark Reader (OMR) answer sheet with black pens, removing the possibility of revising answers later.

CBSE stopped them from carrying pen, pencil, eraser or any electronic gadget and told them to fill the Optical Mark Reader (OMR) answer sheet with black pens, removing the possibility of revising answers later.

In the online version, it doesn't matter whether a student has marked an answer. Unless the applicant clicks on the 'submit answer sheet' button, they can always select/unselect any of the answers.

"We were not informed that pencils and erasers will be barred from the centre. If we had information about the benefits of online exam, we would never have chosen the offline version. This is grave injustice," said Abhilash Sharma, an applicant from Jaipur. The CBSE office in Ajmer did not respond to HT's questions.

Over 1.2 million students appeared for the IIT (JEE) Mains, the first step towards securing a seat in India's premier engineering institutes, the IITs and NITs. A million applicants took the offline examination on 3rd April.

"The paper had three sections with calculation-based numericals. There were a total of 90 questions to be answered in three hours, so it is possible that while solving questions in a hurry, we might have mistakenly darkened the wrong option on the OMR sheet. I wish I had chosen the online exam," said Harshdeep Singh, an applicant.

Aseem Agarwal, the parent of an applicant, wrote to the executive director of the JEE Mains Unit of CBSE to introduce a system in the online examination where the applicants can select an answer only once. But experts said online applicants didn't have an upper hand over offline ones. "CBSE has barred pencils and erasers earlier as well. Also, students were given a choice whether to appear online or offline. I think both have their advantages and disadvantages," said Ashish Arora, academic director of Allen coaching centre.

Courtesy : The Hindustan Times

Math Archives 10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. A bag contains 10 white and 3 black balls. Balls are drawn one by one without replacement till all the black balls are drawn. The probability that the procedure of drawing balls will come to an end at the seventh draw is

(a) $\frac{15}{286}$ (b) $\frac{105}{286}$ (c) $\frac{35}{286}$ (d) $\frac{7}{286}$

2. Let $f : R \to R$ be a differentiable function satisfying $f(y)f(x-y) = f(x) \forall x, y \in R$ and $f'(0) = p$, $f'(5) = q$, then $f'(-5)$ is

(a) $\frac{p^2}{q}$ (b) $\frac{p}{q}$ (c) $\frac{q}{p}$ (d) q

3. The sum of all divisors of the least natural number having 12 divisors is

(a) 168 (b) 188
(c) 156 (d) none of these

4. The number of ordered triplets (p, q, r) where $1 \le p, q, r \le 10$ such that $2^p + 3^q + 5^r$ is a multiple of 4, is $(p, q, r \in N)$

(a) 1000 (b) 500 (c) 250 (d) 125

5. A variable triangle is inscribed in a circle of radius R. If the rate of change of a side is R times the rate of change of the opposite angle, then that angle is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

6. The value of $\lim_{n\to\infty} (\sqrt[3]{n^2 - n^3} + n)$ is

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

7. $ABCD$ and $PQRS$ are two variable rectangles such that A, B, C and D lie on PQ, QR, RS and SP respectively and perimeter 'x' of $ABCD$ is constant. If the maximum area of $PQRS$ is 32, then x is equal to

(a) 8 (b) 10 (c) 12 (d) 16

8. In ΔABC, least value of $\frac{e^A}{A} + \frac{e^B}{B} + \frac{e^C}{C}$ is equal to

(a) $\frac{\pi}{3}e^{\pi/3}$ (b) $\frac{\pi}{9}e^{\pi/3}$
(c) $\frac{9}{\pi}e^{\pi/3}$ (d) none of these

9. The probability that the birthdays of six different persons will fall in exactly two calendar months is

(a) $\frac{341}{12^5}$ (b) $\frac{66}{12^5}$
(c) $\frac{352}{12^5}$ (d) none of these

10 The number of solutions of the equation $\sin^{-1}\left(\frac{1+x^2}{2x}\right) = \frac{\pi}{2}\left(\sec\left(x-1\right)\right)$ is/are

(a) 1 (b) 2 (c) 3 (d) infinite

SOLUTIONS

1. (a) : Required probability = probability of getting exactly two black balls and 4 white balls in 1 to 6th draw × probability of getting 3rd black ball in 7th draw.

$$= \frac{{}^3C_2 \cdot {}^{10}C_4}{{}^{13}C_6} \times \frac{1}{7} = \frac{15}{286}$$

2. (a) : Differentiating the given functional equation with respect to x, we get

$f(y)f'(x-y) = f'(x)$

Putting $x = y$, we get $f'(x) = pf(x)$

$\therefore$ At $x = 0$, $f'(0) = pf(0) \Rightarrow f(0) = 1$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \int p\,dx + C$$

$\Rightarrow \log_e f(x) = px + C$

$\Rightarrow \log_e f(0) = C \Rightarrow C = 0$

B : Prof. Shyam Bhushan, Dire t r, ara ana T adem , am edpur. M b. : 0 33487002

$\Rightarrow f(x) = e^{px} \Rightarrow f'(x) = pe^{px}$

$\therefore \quad f'(5) = pe^{5p} = q \Rightarrow e^{5p} = \frac{q}{p}$

$f'(-5) = pe^{-5p} = \frac{p^2}{q}.$

3. (a) : $12 = 4 \times 3 = 2 \times 2 \times 3$

The number should be of the form $a^1b^1c^2$ where a, b, c are prime numbers.

For number to be least

$c = 2, b = 3, a = 5$

Least number $= 2^2\ 3^1\ 5^1$

Sum of all divisors

$= (2^0 + 2^1 + 2^2)(3^0 + 3^1)(5^0 + 5^1)$

$= (7)(4)(6) = 168.$

4. (b) : $2^p + 3^q + 5^r = 2^p + (4 - 1)^q + (4 + 1)^r$

$= 2^p + 4\lambda_1 + (-1)^q + 4\lambda_2 + 1^r$ (λ_1, λ_2 are integers)

If $p = 1$, q should be even and r can be any number. On the other hand if $p \neq 1$, q should be odd and r can be any number.

Total number of ordered triplets

$= 5 \times 10 + 9 \times 5 \times 10 = 500.$

5. (c) : Let side $BC = a$ and A be opposite angle

$\therefore \quad R = \frac{a}{2\sin A} \quad \Rightarrow \quad a = 2R\sin A$

Now, $\frac{da}{dt} = 2R\cos A\frac{dA}{dt}$

$\Rightarrow \quad R\frac{dA}{dt} = 2R\cos A\frac{dA}{dt} \qquad \left[\because \frac{da}{dt} = \frac{RdA}{dt}\right]$

$\Rightarrow \quad \cos A = \frac{1}{2} \quad \Rightarrow \quad A = \frac{\pi}{3}$

6. (a) : $\lim\limits_{n\to\infty}\left(n\sqrt[3]{\frac{1}{n}-1}+n\right)$

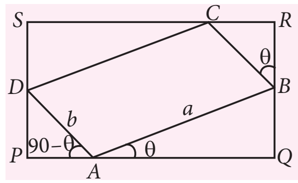
$$= \lim_{n\to\infty} n\left[\left(\frac{1}{n}-1\right)^{1/3}+1^{1/3}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\left(\frac{1}{n}-1\right)+1}{\left(\frac{1}{n}-1\right)^{2/3}-\left(\frac{1}{n}-1\right)^{1/3}+1}\right]$$

$$\left\{a+b = \frac{a^3+b^3}{a^2-ab+b^2}\right\}$$

$$= \lim_{n\to\infty}\left[\frac{1}{\left(\frac{1}{n}-1\right)^{2/3}-\left(\frac{1}{n}-1\right)^{1/3}+1}\right] = \frac{1}{3}$$

7. (d) : Here $2a + 2b = x \quad \Rightarrow \quad a + b = \frac{x}{2}$



Area (A) of rectangle $PQRS = PQ \times QR$

$= (PA + AQ) \cdot (QB + BR)$

$= (b\sin\theta + a\cos\theta)(a\sin\theta + b\cos\theta)$

$$= ab + \frac{(a^2+b^2)\sin 2\theta}{2}$$

$$A \le ab + \frac{a^2+b^2}{2} \quad \text{or} \quad A \le \frac{(a+b)^2}{2}$$

$$A_{max} = \frac{(a+b)^2}{2} = 32 \quad \Rightarrow \quad \frac{x^2}{8} = 32$$

$\therefore \quad x = 16.$

8. (c) : We have, $\frac{e^A}{A}+\frac{e^B}{B}+\frac{e^C}{C} \ge 3\left(\frac{e^{A+B+C}}{ABC}\right)^{1/3}$

Again $A + B + C \ge 3(ABC)^{1/3} \Rightarrow \frac{\pi}{3} \ge (ABC)^{1/3}$

$$\Rightarrow \quad \left(\frac{1}{ABC}\right)^{1/3} \ge \frac{3}{\pi}$$

$$\Rightarrow \quad \left(\frac{e^\pi}{ABC}\right)^{1/3} \ge \frac{3}{\pi}e^{\pi/3} \quad \Rightarrow \quad 3\left(\frac{e^\pi}{ABC}\right)^{1/3} \ge \frac{9}{\pi}e^{\pi/3}$$

$$\Rightarrow \quad \frac{e^A}{A}+\frac{e^B}{B}+\frac{e^C}{C} \ge \frac{9}{\pi}e^{\pi/3}$$

9. (a) : Total number of ways in which 6 persons can have their birthdays $= 12^6$

Out of 12 months, 2 months can be chosen in ${}^{12}C_2$ ways. Now birthdays of six persons can fall in these two months in 2^6 ways. Out of these 2^6 ways, there are two ways when all six birthdays fall in one month.

So there are $(2^6 - 2)$ ways in which six birthdays fall in chosen 2 months.

$$\text{Required probability} = \frac{{}^{12}C_2\left(2^6-2\right)}{12^6} = \frac{341}{12^5}$$

10. (a) : We have, $\sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined for

$$\left|\frac{1+x^2}{2x}\right| \le 1 \quad \Rightarrow \quad x = \pm 1$$

Out of these two values of x, only $x = 1$ satisfies the given equation.

Rvp